

RECONSTRUCTION OF STRATIFIED STEADY WATER WAVES FROM PRESSURE READINGS ON THE OCEAN BED

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ABSTRACT. Consider a two-dimensional stratified solitary wave propagating through a body of water that is bounded below by an impermeable ocean bed. In this work, we study how such a wave can be reconstructed from data consisting of the wave speed, upstream and downstream density profile, and the trace of the pressure on the bed. First, we prove that this data uniquely determines the wave, both in the (real) analytic and Sobolev regimes. Second, for waves that consist of multiple layers of constant density immiscible fluids, we provide an exact formula describing each of the interfaces in terms of the data. Finally, for continuously stratified fluids, we detail a reconstruction scheme based on approximation by layer-wise constant density flows.

1. INTRODUCTION

One of the primary means of measuring waves at sea are *pressure transducers* — recording devices distributed throughout the ocean that read the subsurface pressure. There are many advantages to this approach: pressure transducers consume relatively little energy, are inexpensive to build and maintain, and provide a wealth of *in situ* data. As they are deployed submerged, moreover, they have a minimal impact on marine life and ocean traffic.

The use of pressure transducers raises several intriguing mathematical questions: How much information can really be gleaned about the flow from just the trace of the pressure on the ocean bed? Is it possible to totally reconstruct a passing traveling wave from this data? These issues are the subject of the present paper. Specifically, we are concerned with the role density stratification may play and whether pressure transducers can be used to track internal waves.

Let us first discuss the situation for constant density fluids. If one takes the linearized free surface Euler equations for a small-amplitude gravity wave, it is simple to show that the surface profile can be expressed in terms of the pressure trace; indeed, this mapping is a Fourier multiplier whose symbol is referred to as the (linear) *pressure transfer function* (see, e.g., [6, 18, 25, 27]). More concretely, for a sinusoidal wave with wavenumber k , the connection between the the dynamic pressure \mathfrak{p} and the surface displacement η can be written as

$$(1.1) \quad \mathfrak{p} = \rho g \frac{\cosh(kh_m)}{\cosh(kd)} \eta,$$

where h_m is the height of the pressure meter above the bottom, d is the mean depth, and the dynamic pressure \mathfrak{p} is defined to be the difference between the pressure and the hydrostatic pressure, that is,

$$\mathfrak{p} = P - P_{\text{atm}} - \rho g(d - h_m).$$

For pressure data obtained from the sea-bed, $h_m = 0$ and hence (1.1) becomes

$$\mathfrak{p} = \frac{\rho g}{\cosh(kd)} \eta,$$

which, in the zero depth limit $d \rightarrow 0$, recovers the hydrostatic approximation

$$(1.2) \quad \mathfrak{p} = \rho g \eta.$$

However, it is well known that nonlinear effects play an important role in the surface reconstruction problem for shallow water waves or for waves in the surf zone (see [5, 6, 39], for instance). Experiments suggest that the predictions of linear models such as (1.1) may diverge sharply from observation [7]. For waves of large or even moderate amplitude, the discrepancy can be significant [6, 39]. Another issue with the linear theory is that, due to its reliance on Fourier series techniques, it can only treat periodic traveling waves.

These limitations have motivated a recent push to obtain reconstruction formulas that account for nonlinear effects and which apply to solitary waves. Nonlinear nonlocal equations relating the trace of the dynamic pressure on the bed to the surface profile of a solitary wave were established by several groups of researchers (see [11, 13, 31]). Notably, these papers work directly from the free surface Euler equations without further approximation. In [9, 10], the authors obtained implicit but exact and tractable relations for periodic traveling waves and performed straightforward numerical procedures to derive the free surface from the pressure at the bed. In [12], Constantin used a strong maximum principle argument completely different from the aforementioned approach to derive some improved estimates for the wave height of periodic traveling waves.

The above body of work focuses on irrotational steady waves propagating through constant density water. Actual waves in the ocean, however, frequently exhibit a heterogeneous density due to salinity or temperature gradients. This density stratification generates vorticity within the flow as the effects of gravity and inertia are experienced differently by heavier and lighter fluid particles. Moreover, a heterogeneous density distribution can permit large-amplitude waves to propagate in the bulk even while the surface remains relatively undisturbed. These so-called *internal waves* have been observed in oceanic areas for many years (cf., e.g. [33], or the compendium in [23]). They are known to have important implications for particle transport [24, 36, 37], mixing and energy dissipation [22, 26, 29, 30], as well as affecting acoustic propagation [34, 35, 50].

The vorticity that naturally accompanies stratification poses a serious mathematical challenge: nearly all of the irrotational theory is built on tools (e.g., conformal mappings, nonlocal reformulations via Dirichlet–Neumann operators, variational principles, etc.) that do not have obvious analogues in the rotational regime. As a consequence, rigorous results on rotational waves have begun to appear in the literature only quite recently. Assuming the vorticity is constant throughout the fluid domain, which is the simplest model of a wave-current interaction, Ali and Kalisch [2] derived a relationship in the long-wave limit that gives a direct map between the surface elevation and the pressure at the bottom of the fluid and within the bulk of the fluid domain. Later Visan and Oliveras [42] obtained an exact reconstruction formula in this regime without approximation.

At present, no one to our knowledge has discovered how to adapt these ideas to the case of an *arbitrary* vorticity distribution. Indeed, constant vorticity is a very special (though physically significant) case that rarely provides much insight into how to attack the general problem. However, in [19], D. Henry proved that, if the vorticity and pressure are real analytic, then the trace

of the pressure on the bed uniquely determines a solitary rotational wave. The analyticity condition is needed because he employs a Cauchy-Kowalevski argument. In that sense, the surface reconstruction problem is well-defined even for rotational waves so long as they are extremely regular.

Unique determinability with stratification. Our first results pertain to the question of unique determinability of a stratified steady wave from the trace of the pressure on the bed. Mathematically, stratification plays a somewhat similar role in the governing equation for steady waves as vorticity. This suggests that Henry's method can be adapted to analytically stratified flows. Following this strategy, we prove that, if two stratified traveling waves share the same wave speed, analytic streamline density function, and analytic pressure trace on the bed, then they must coincide exactly; see Theorem 3.1 and Corollary 3.1.

However, analyticity is a very strong assumption that is particularly ill-suited to stratified waves. Water columns in the ocean are frequently observed to have nearly constant density outside of thin transition layers called pycnoclines. Thus, as it passes through a pycnocline, the density experiences something close to a jump discontinuity. Indeed, a common practice is to view these waves as being a layering of multiple immiscible fluids each with its own constant density. In this model, the density distribution is piece-wise analytic, allowing us to iterate the Cauchy-Kowalevski approach of Henry and determine all of the free surfaces (cf. Corollary 3.2).

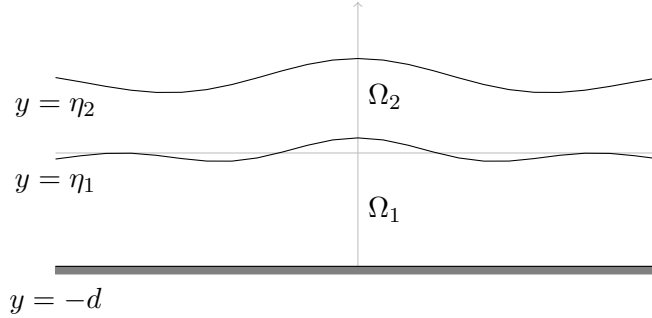
The more interesting question is whether one has unique determinability for a continuous but not necessarily analytic density (e.g., the existence theory in [44] is formulated for density of class $C^{1,\alpha}$). For this case, Henry's argument does not seem to apply and so an entirely new idea is needed. In Theorem 3.4, we confirm that unique determinability still holds in the Sobolev space setting. Our method is based on reformulating the problem as an elliptic equation on a strip which can be treated using the theory of strong unique continuation (cf. Theorem 3.2). In fact, this approach can be used for constant density rotational waves as well, hence Theorem 3.4 generalizes Henry's work to the Sobolev regularity regime (see Remark 3.2).

Surface reconstruction. The results above show that surface reconstruction from pressure problem is mathematically well-defined for stratified waves, even those with a more realistic degree of smoothness. As the second contribution of this paper, we present an explicit scheme for performing this reconstruction. In order to explain our ideas, first we must discuss the physical setup in more detail.

We are interested in two-dimensional solitary waves with heterogeneous density ϱ , limiting to uniform flows upstream and downstream, and traveling with a constant speed c . Shifting to a moving reference frame eliminates time dependence from the system and thus we may assume the wave occupies a steady fluid domain

$$(1.3) \quad \Omega = \{(x, y) \in \mathbb{R}^2 : -d < y < \eta(x)\},$$

where the unknown function η is the free surface profile. We call the fluid *continuously stratified* provided that $\varrho \in C(\overline{\Omega})$. We also assume that $\varrho > 0$ in $\overline{\Omega}$ and that the fluid is stably stratified, i.e., $y \mapsto \varrho(\cdot, y)$ is non-increasing.

FIGURE 1. Example with two fluid layers, Ω_1 and Ω_2 .

Continuously stratified fluids are difficult to analyze directly since they are rotational. We therefore begin by studying the *layered model* wherein Ω is partitioned into finitely many immiscible fluid regions

$$(1.4) \quad \Omega = \bigcup_{i=1}^N \Omega_i, \quad \Omega_i := \{(x, y) \in \Omega : \eta_{i-1}(x) < y < \eta_i(x)\}$$

with the density $\varrho_i := \varrho|_{\Omega_i}$ constant in each layer. Here we index so that $\eta_0 := -d$, $\eta_N := \eta$ and Ω_i lies beneath Ω_{i+1} , for $i = 1, \dots, N-1$; see Figure 1. Note that this is implicitly assuming that the free surfaces all have graph geometry.

The first task is to develop a procedure that allows us to recover a layer-wise constant density irrotational solitary waves from its pressure trace on the bed. For this, we can exploit the fact that the velocity potential φ and standard stream function ψ_s are harmonic conjugates within each layer Ω_i . Since the density is constant, φ and the *pseudo-stream function* $\psi := \sqrt{\varrho_i} \psi_s$ are harmonic conjugates as well. The pseudo-stream function is better adapted to the study of stratified flows than the standard stream function. Then, the conformal change of variables

$$(x, y) \mapsto \left(-\frac{1}{c} \varphi(x, y), -\frac{1}{c} \psi(x, y) \right),$$

transforms each layer Ω_i to a fixed horizontal strip (cf. Section 2.4). At this point, an adaptation of Constantin's method [11] allows us to reconstruct the flow in the first layer up to the first interface. However, when we attempt to continue the process and compute the flow in the second layer, we are confronted with an unfortunate feature of the conformal coordinates: because the velocity potential is in general discontinuous over the interface, the conformal variables behave badly as one crosses from one fluid region to the next. This necessitates some additional work, but can be overcome with a further reparameterization of the data on the interface, and by paying careful attention to the jump conditions there (cf. Section 4.5). Ultimately, we are able to obtain an explicit reconstruction that determines the flow in every layer, as well as all of the free surfaces, directly from the trace of the pressure on the bed.

Finally, we return to the case of continuously stratified waves using an approximation argument. As discussed above, the layered model is a convenient idealization of a continuously stratified fluid. In a very recent work, the authors showed that it can in fact be made *rigorous* for periodic traveling

waves in a certain small-amplitude regime (see [8]). That is, one can show that as $\sum_{i=1}^N \varrho_i \mathbf{1}_{\Omega_i}$ converges to ϱ in L^∞ , the corresponding layer-wise smooth waves converge to the continuously stratified one. On the other hand, solitary waves in this same regime can be realized as the limit of periodic traveling waves taking the period to infinity [41]. Using a diagonalization argument, we are therefore able to reconstruct a continuously stratified solitary wave as a limit of layer-wise constant waves considered in the previous paragraph; see the discussion in Section 5.2.

2. FORMULATION

In this section, we present several formulations of the steady stratified water wave system, each one tailored to one of the problems we later consider. Their equivalence is fairly straightforward to show (cf., e.g., [8, Lemma A.2] where it is proved for much weaker regularity solutions).

2.1. Eulerian formulation. Consider a two-dimensional wave in a stratified water propagating to the right with speed $c > 0$. We assume that the only restorative force is gravity. Adopting a moving reference frame, we can view the system as time-independent. The fluid domain Ω is a priori unknown, but, as in (1.3), we assume that it lies below the graph of some smooth function η and above an impermeable ocean bed at $\{y = -d\}$.

Let us now formulate the governing equations for a multi-layered wave, which includes the family of continuously stratified fluid as a special case. With that in mind, suppose that Ω is organized into immiscible strata as in (1.4). Then the wave is described mathematically by a velocity field $(u, v) : \Omega \rightarrow \mathbb{R}^2$, a density function $\varrho : \Omega \rightarrow \mathbb{R}_+$, and a pressure $P : \Omega \rightarrow \mathbb{R}$. We follow the convention that, for a quantity f with domain Ω , $f_i := f|_{\Omega_i}$.

We are interested in (u, v, ϱ) for which each restriction (u_i, v_i, ϱ_i) is reasonably smooth in each $\overline{\Omega_i}$, but we do not impose any continuity assumptions on $\overline{\Omega}$. By contrast, P will be continuous throughout the fluid domain. Note that at this stage we do not require that ϱ_i is a constant.

To represent a traveling water wave with wave speed c , (u, v, ϱ, P) must solve the incompressible Euler system: conservation of mass

$$(2.1a) \quad (u - c)\varrho_x + v\varrho_y = 0, \quad \text{in } \Omega_i,$$

conservation of momentum

$$(2.1b) \quad \varrho(u - c)u_x + \varrho v u_y = -P_x, \quad \text{in } \Omega_i,$$

$$(2.1c) \quad \varrho(u - c)v_x + \varrho v v_y = -P_y - g\varrho, \quad \text{in } \Omega_i,$$

and incompressibility

$$(2.1d) \quad u_x + v_y = 0, \quad \text{in } \Omega_i.$$

Here g the gravitational constant of acceleration. We say a fluid is stably stratified if

$$(2.2) \quad y \mapsto \varrho(\cdot, y) \text{ is non-increasing.}$$

The motion of the interfaces is driven by the kinematic boundary condition:

$$(2.3a) \quad v = (u - c)\partial_x \eta_i, \quad \text{on } \{y = \eta_i(x)\},$$

Note that this includes the air-sea interface. Also implicit above is the fact that

$$\frac{v_i}{u_i - c} = \frac{v_{i+1}}{u_{i+1} - c} \quad \text{on } \{y = \eta_i(x)\},$$

which follows from the continuity of the normal component of the velocity over the interface. Since the ocean bed is impermeable, we must have that

$$(2.3b) \quad v = 0, \quad \text{on } \{y = -d\}.$$

Finally, for the pressure we require

$$(2.3c) \quad P = P_{\text{atm}}, \quad \text{on } \{y = \eta(x)\},$$

where P_{atm} is the (constant) atmospheric pressure. This choice effectively neglects surface tension.

It will be important for our later reformulations to assume that there is no horizontal stagnation in the flow:

$$(2.3d) \quad u - c < 0 \quad \text{in } \overline{\Omega}.$$

Looking at (2.1b)–(2.1c), it is clear that (2.3d) prohibits a certain degeneracy in the system.

In this work, we mostly consider solitary waves. This means that we impose the asymptotic conditions

$$(2.4) \quad (u, v) \rightarrow (\dot{u}, 0), \quad \eta(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

Here $\dot{u} = \dot{u}(y)$ is a given function. In most cases, we will assume that $\dot{u} = 0$, i.e., there is uniform velocity at infinity. For the time being, though, we state things in their full generality.

2.2. Stream function formulation. The Euler equations are a complicated nonlinear system. It is often far more convenient to reformulate them in terms of a scalar quantity. Observe that the conservation of mass (2.1a) and incompressibility (2.1d) ensure that we may define a function $\psi = \psi(x, y)$ by

$$(2.5) \quad \psi_x = -\sqrt{\varrho}v, \quad \psi_y = \sqrt{\varrho}(u - c) \quad \text{in } \Omega.$$

This is the pseudo (relative) stream function for the flow, but throughout this paper we will simply refer to it as the *stream function*. From (2.5) and (2.3d), we see that the no stagnation condition translates to

$$(2.6) \quad \psi_y < 0 \quad \text{in } \Omega.$$

The level sets of ψ are called the *streamlines*. In fact (2.3a)–(2.3b) directly imply that the free surface, internal interfaces, and ocean bed are all streamlines. Since (2.5) only determines ψ up to a constant in each Ω_i , we may take ψ to be continuous in $\overline{\Omega}$, and set $\psi = 0$ on the air-sea interface. Then $\psi = -p_0$ on the bed $\{y = -d\}$, where p_0 is the (*relative*) *pseudo-volumetric mass flux*:

$$(2.7) \quad p_0 := \int_{-d}^{\eta(x)} \sqrt{\varrho(x, y)} [u(x, y) - c] dy,$$

which is a (strictly negative) constant. Likewise, we define

$$(2.8) \quad p_i := \int_{-d}^{\eta_i(x)} \sqrt{\varrho(x, y)} [u(x, y) - c] dy,$$

so that $\{y = \eta_i(x)\}$ coincides with the level set $\{\psi = -p_i\}$.

The conservation of mass (2.1a) implies that there exists a function $\rho : [p_0, 0] \rightarrow \mathbb{R}^+$ such that

$$(2.9) \quad \varrho(x, y) = \rho(-\psi(x, y)) \quad \text{in } \Omega.$$

We refer to ρ as the *streamline density function* and treat it as given. For solitary waves, one determines ρ by examining the flow upstream or downstream.

Conservation of energy can be expressed via Bernoulli's theorem, which states that the quantity

$$(2.10) \quad E := P + \frac{\varrho}{2} ((u - c)^2 + v^2) + g\varrho y,$$

is constant along streamlines. This allows us to define the so-called *Bernoulli function* $\beta : [0, |p_0|] \rightarrow \mathbb{R}$ by

$$(2.11) \quad \frac{dE}{d\psi}(x, y) = -\beta(\psi(x, y)), \quad \text{in } \Omega.$$

The dynamics in the interior of each layer is then captured by Yih's equation [49]

$$(2.12a) \quad \Delta\psi - g\varrho'(-\psi) + \beta(\psi) = 0 \quad \text{in } \Omega.$$

Note that if in some layer Ω_i the flow is irrotational and ρ_i is a constant, then (2.12a) reduces to Laplace's equation for ψ .

Lastly, for the stream function formulation to be equivalent to the full Euler system, it is necessary that E satisfies certain jump conditions on the free surfaces. On the air-sea interface, $P = P_{\text{atm}}$ and hence (2.10) gives immediately that

$$(2.12b) \quad |\nabla\psi|^2 + 2g\varrho_N(y + d) = Q, \quad \text{on } \{y = \eta(x)\},$$

where $Q = 2(E + \varrho d)|_{y=\eta}$. Likewise, for the interior interfaces we use the fact that P is continuous in $\overline{\Omega}$ when evaluating E from above and below on $\{y = \eta_i(x)\}$. This results in

$$(2.12c) \quad |\nabla\psi_{i+1}|^2 - |\nabla\psi_i|^2 + 2g(\rho_{i+1} - \rho_i)(\eta_i + d) = Q_i \quad \text{on } \{y = \eta_i(x)\},$$

with $Q_i := 2(E_{i+1} + g\rho_{i+1} - E_i - g\rho_i)|_{y=\eta_i}$.

For solitary waves, the form at infinity strongly constrains the global structure. First, observe that (2.4) implies that

$$(2.12d) \quad \nabla^\perp\psi = \sqrt{\varrho}(u - c, v) \rightarrow \sqrt{\rho}(\hat{u} - c, 0), \quad \eta(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

Let us focus on the simplest case where $\hat{u} = 0$. Then, letting $\mathring{h}(p)$ denote the limiting height above the bed of the streamline $\{\psi = -p\}$, from (2.5) we have

$$(2.12e) \quad \mathring{h}(p) = \int_{p_0}^p \frac{1}{c\sqrt{\rho(s)}} ds.$$

In fact, this couples the wave speed and density because we can evaluate the above identity on the air–sea interface to get

$$(2.12f) \quad d = \int_{p_0}^0 \frac{1}{c\sqrt{\rho(s)}} ds.$$

Now, it follows that

$$(2.12g) \quad Q = \varrho_N c^2 + 2g\varrho_N d, \quad Q_i = (\varrho_{i+1} - \varrho_i) (c^2 + 2g\mathring{h}).$$

Finally, taking the upstream limit of (2.10), we can actually compute β directly in terms of \mathring{h} :

$$(2.12h) \quad \beta(p) = \rho'(p) \left[\frac{1}{2}c^2 + g(\mathring{h} - d) \right].$$

This is a well known fact about solitary stratified waves, but an elementary derivation is given in [8].

In summary, the stream function formulation of the problem is to find ψ continuous in $\overline{\Omega}$ that satisfies Yih's equation (2.12a) in each Ω_i , along with Bernoulli boundary condition (2.12b) on the air–sea interface, the jump condition (2.12c) on the internal interfaces, and the no stagnation condition (2.3d). In addition, for a solitary wave, we must have that ψ exhibits the asymptotic behavior (2.12d), which in turn implies that the constant Q , Q_i are given as in (2.12g).

2.3. Height function formulation. The stream function formulation has the advantage of being a scalar equation, but it is still posed in an a priori unknown domain. For that reason, it is useful to consider changes of variables that map Ω to a fixed domain. Of course, one should expect this to come at the cost of additional nonlinearity in the resulting PDE. A particularly convenient choice is to adopt semi-Lagrangian variables

$$(x, y) \mapsto (q, p) := (x, -\psi).$$

These coordinates have a long history in the literature on existence of stratified water waves; their use dates back to Dubreil-Jacotin [16]. Note that, because the internal interfaces and free surface are level sets of ψ , each layer Ω_i maps to a strip $\mathcal{R}_i := \{(q, p) : p \in (p_{i-1}, p_i)\}$. Similarly, $\mathcal{R} := \bigcup_i \mathcal{R}_i$ is the image of Ω .

Let

$$h = h(q, p) := y + d,$$

which denotes the height above the bed of the point with x -coordinate q that lies on the streamline $\{\psi = -p\}$. Notice that h is continuous in $\overline{\mathcal{R}}$, though its derivatives will in general not be.

One can show that Yih's equation (2.12a) is equivalent to the quasilinear elliptic problem

$$(2.13a) \quad (1 + h_q^2)h_{pp} + h_{qq}h_p^2 - 2h_qh_ph_{pq} - g(h - d)\rho_ph_p^3 = -h_p^3\beta(-p) \quad \text{in } \mathcal{R}.$$

The Bernoulli condition on the top (2.12b) likewise becomes

$$(2.13b) \quad 1 + h_q^2 + h_p^2(2g\rho_h - Q) = 0 \quad \text{on } \{p = 0\},$$

while the jump conditions on the internal interfaces translate to

$$(2.13c) \quad \frac{1 + (\partial_q h_{i+1})^2}{(\partial_p h_{i+1})^2} - \frac{1 + (\partial_q h_i)^2}{(\partial_p h_i)^2} + 2g(\rho_{i+1} - \rho_i)h = Q_i \quad \text{on } \{p = p_i\}.$$

Here we are adapting our previous convention and writing $h_i := h|_{\mathcal{R}_i}$. Finally, simply from the definition it follows that

$$(2.13d) \quad h = 0 \quad \text{on } \{p = p_0\}.$$

The derivation of this system can be found in many places (see, e.g., [44, 47]).

Restricting our attention to solitary waves with uniform velocity at infinity, (2.13) can be simplified by taking β of the form (2.12h), and Q, Q_i as in (2.12g). For convenience, we record here the resulting system for a continuously stratified fluid (the layer-wise smooth case being a straightforward generalization):

$$(2.14) \quad \begin{cases} (1 + h_q^2)h_{pp} + h_{qq}h_p^2 - 2h_qh_ph_{pq} = -h_p^3\rho_p \left[\frac{1}{2}c^2 + g(\mathring{h} - h) \right], & \text{in } \mathcal{R}, \\ 1 + h_q^2 + h_p^2(2g\rho h - Q) = 0, & \text{on } \{p = 0\}, \\ h = 0, & \text{on } \{p = p_0\}. \end{cases}$$

Naturally, the definition of \mathring{h} (2.12e) implies that we also have

$$(2.15) \quad h \rightarrow \mathring{h} \quad \text{as } |q| \rightarrow \infty.$$

2.4. Reformulation using conformal coordinates. Recall that ψ is given by (2.5) along with the proscription that ψ is continuous on $\overline{\Omega}$. It follows that ψ can be written

$$\psi(x, y) = -p_0 + \int_{-d}^y \sqrt{\rho(x, z)} [u(x, z) - c] \, dz.$$

From this expression, it is easy to confirm that $\psi \in C^{0,1}(\overline{\Omega})$. Observe also that Yih's equation (2.12a) implies that, if β takes the form (2.12h), then in any layer where ϱ_i is constant, the corresponding velocity field is irrotational and ψ_i is harmonic.

We may therefore define a harmonic function φ with domain $\bigcup_i \Omega_i$ by taking $\varphi|_{\Omega_i}$ to be a harmonic conjugate of $\psi|_{\Omega_i}$. Thus $\nabla\varphi = \sqrt{\rho}(u - c, v)$, i.e., φ is a pseudo velocity potential in each layer. This only determines $\varphi|_{\Omega_i}$ up to a constant. Anticipating an imminent change of variables, we choose to normalize so that

$$(2.16) \quad \varphi(0, -d) = 0 \quad \text{and} \quad \{(x, y) \in \overline{\Omega} : \varphi(x, y) = 0\} \text{ is a connected set.}$$

Note that there exists a unique continuous extension of $\varphi|_{\Omega_i}$ to $\overline{\Omega_i}$, but in general φ will not be continuous over the interior interfaces. This can be viewed as a consequence of the fact that the tangential velocity may jump between layers.

As φ and ψ are harmonic conjugates in each Ω_i , one can employ the hodograph transform:

$$\mathcal{H} : (x, y) \mapsto \left(-\frac{1}{c}\varphi(x, y), -\frac{1}{c}\psi(x, y) \right) =: (t, s).$$

Then the image of each fluid layer Ω_i is a strip

$$\mathcal{T}_i := \{(t, s) \in \mathbb{R} \times (s_{i-1}, s_i)\}$$

with $s_i := p_i/c$. The fluid domain Ω is thus mapped to an N -slitted strip $\mathcal{T} := \bigcup_i \mathcal{T}_i$.

To keep the various formulations clear, designate the height above the bed in the (t, s) -coordinates by H :

$$H(t, s) = y + d \quad \text{for } (t, s) \in \mathcal{T}.$$

Therefore the i -th interface $\{y = \eta_i(x)\}$ can be parametrized by t :

$$(2.17) \quad Y_i(t) = H_i(t, s_i) - d.$$

In order to return to the physical variables (x, y) , it is also necessary to find a parameterization $x_i = x_i(t, s_i)$ in a way such that

$$(2.18) \quad (x, \eta_i(x)) = (x_i(t, s_i), Y_i(t)).$$

To reformulate the problem in the conformal coordinates, we begin collecting some simple change of variable calculations:

$$(2.19) \quad \begin{cases} \partial_x = \frac{\sqrt{\varrho_i}}{c} [(c-u)\partial_t + v\partial_s], \\ \partial_y = \frac{\sqrt{\varrho_i}}{c} [-v\partial_t + (c-u)\partial_s], \end{cases} \quad \begin{cases} H_t = y_t = -\frac{cv}{\sqrt{\varrho_i}[(c-u)^2 + v^2]} = -x_s, \\ H_s = y_s = \frac{c(c-u)}{\sqrt{\varrho_i}[(c-u)^2 + v^2]} = x_t, \end{cases} \quad \text{in } \mathcal{T}_i.$$

From this it also follows that

$$(2.20) \quad \begin{cases} \partial_t = H_s \partial_x + H_t \partial_y, \\ \partial_s = -H_t \partial_x + H_s \partial_y, \end{cases} \quad \begin{cases} v = -\frac{cH_t}{\sqrt{\varrho_i}(H_t^2 + H_s^2)}, \\ c-u = \frac{cH_s}{\sqrt{\varrho_i}(H_t^2 + H_s^2)}, \end{cases} \quad \text{in } \mathcal{T}_i.$$

Therefore the height function satisfies

$$(2.21) \quad \begin{cases} \Delta_{(t,s)} H = 0, & \text{in } \mathcal{T}, \\ H = 0, & \text{on } s = s_0, \\ (H_t^2 + H_s^2)(c^2 - 2gH + 2gd) = \frac{c^2}{\varrho_N^2} & \text{on } s = 0. \end{cases}$$

For localized solitary waves as considered in [3, 4, 38, 40, 41], we have the following asymptotics

$$(2.22) \quad \begin{cases} \lim_{|t| \rightarrow \infty} \nabla_{(t,s)} H = \left(0, \frac{1}{\sqrt{\varrho_i}}\right) & \text{uniformly for } s \in (s_{i-1}, s_i). \\ \lim_{|t| \rightarrow \infty} H(t, s_i) = \mathring{H}(s_i), & i = 1, \dots, N, \end{cases}$$

where $\mathring{H}(s) = \mathring{h}(cs)$. Moreover using (2.12e) to evaluate \mathring{H} at s_i yields

$$(2.23) \quad \mathring{H}(s_i) = \int_{p_0}^{p_i} \frac{1}{c\sqrt{\rho(\tau)}} d\tau = \frac{1}{c} \sum_{j=0}^{i-1} \frac{p_{j+1} - p_j}{\sqrt{\varrho_{j+1}}} = \sum_{j=0}^{i-1} \frac{s_{j+1} - s_j}{\sqrt{\varrho_{j+1}}}.$$

3. DETERMINABILITY

The objective of this section is to establish that the pressure reconstruction problem is mathematically well-defined. That is, we prove in various regimes that the wave speed, streamline density function, Bernoulli function, and the trace of the pressure on the bed together uniquely determine a steady stratified water wave. In particular, among all solitary waves with a given form at infinity, to each pressure trace there corresponds precisely one surface profile.

The main insight underlying the argument is that, due to Bernoulli's law, the trace of the pressure on the bed gives Neumann data for the height function. Since we already know that the height is 0 on the bed, together this amounts to Cauchy data.

To see this, first observe that, in the semi-Lagrangian coordinates, Bernoulli's law (2.10) becomes

$$(3.1) \quad E = P + \frac{1 + h_q^2}{2h_p^2} + g\rho(h - d).$$

Evaluating this on the flat bed and using the definition of the Bernoulli function β (2.11) leads to

$$(3.2) \quad E|_{y=-d} = E|_{\psi=p_0} = E|_{\psi=0} - \int_0^{p_0} \beta(p) \, dp.$$

As before, the form of the flow in the far field (2.4) allows us to compute that the Bernoulli constant $E|_{\psi=0}$ is

$$(3.3) \quad E|_{\psi=0} = E|_{y=\eta} = P_{\text{atm}} + \frac{1}{2}\rho(0)(c - \dot{u}(0))^2.$$

Therefore, from (2.13d) and (3.1)–(3.3), we infer that

$$P(q, p_0) + \frac{1}{2h_p^2(q, p_0)} - g\rho(p_0)d = P_{\text{atm}} + \frac{1}{2}\rho(0)(c - \dot{u}(0))^2 - \int_0^{p_0} \beta(p) \, dp.$$

Finally, the absence of stagnation (2.3d) implies $h_p > 0$, so we conclude

$$(3.4) \quad h_p(q, p_0) = \left[2 \left(P_{\text{atm}} + \frac{1}{2}\rho(0)(c - \dot{u}(0))^2 + g\rho(p_0)d - \int_0^{p_0} \beta(p) \, dp - P(q, p_0) \right) \right]^{-1/2}.$$

Thus the normal derivative of h on $\{p = p_0\}$ is known as soon as the streamline density ρ , Bernoulli function β , limiting downstream/upstream horizontal velocity on the free surface, and the pressure data on the sea-bed are specified.

3.1. Analytic setting. We have seen that, treating the pressure trace as given, (2.13d) and (3.4) amount to Cauchy data. It is therefore useful to transform the elliptic height equation (2.13) into a first-order system that can be approached via Cauchy–Kowalevski. Define

$$(3.5) \quad F := \frac{h_q}{h_p}, \quad G := \frac{1}{h_p}, \quad H := h.$$

This is an abuse of notation in the sense that H above is not the same as the one appearing in Section 2.4. Nonetheless, for the purposes of the discussion in this subsection, there is little risk of confusion.

From (2.3d) and the identity $1/h_p = \sqrt{\rho}(c - u)$ we know that these quantities are well-defined. We can then reformulate the height equation (2.13) as

$$(3.6) \quad \begin{cases} F_p = \frac{F}{G} \left[\frac{GF_q + FG_q + G\beta(-p) - Gg\rho_p(H-d)}{F^2 + G^2} \right] - \frac{G_q}{G}, \\ G_p = \frac{GF_q + FG_q + G\beta(-p) - Gg\rho_p(H-d)}{F^2 + G^2}, \\ H_p = \frac{1}{G}, \end{cases}$$

with the following “initial” conditions posed at $p = p_0$

$$F(q, p_0) = H(q, p_0) = 0.$$

Here $G(\cdot, p_0)$ is uniquely determined by the pressure trace $P(\cdot, p_0)$ via (3.4).

Theorem 3.1. *Let $h \in C^{2,\alpha}(\overline{\mathcal{R}})$ be a solution of the height equation (2.14) representing a solitary water wave with a given stable real analytic streamline density stratification, a real analytic Bernoulli function, wave speed c , and downstream/upstream horizontal velocity on the surface. Assume that the wave speed exceeds the horizontal fluid velocity $u < c$ through out the fluid. Then h is real analytic and is uniquely determined by the pressure function P on the flat bed.*

Proof. The Cauchy-Kowalevski theorem applied to system (3.6) implies the existence and uniqueness of F, G, H , which are real analytic in some open neighborhood \mathcal{N} of the bed $\{p = p_0\}$. Hence, in view of (3.5), h, h_p, h_q are also real analytic in \mathcal{N} . On the other hand, under the regularity hypotheses above, all solutions of the height equation (2.13) are real analytic in $\overline{\mathcal{R}}$ (cf. [48, Theorem 5.1]). Therefore the unique continuation of real analytic functions ensures the uniqueness of the solution h to (2.14) with Neumann data on the bottom given by the pressure from (3.4). \square

In contrast to the case of solitary waves, a fairly robust existence theory for large-amplitude periodic steady stratified waves has been developed in recent years (cf. [44, 17, 21, 20, 45, 46]). Since the Cauchy-Kowalevski theory is local, it can be applied just as well on a periodic domain. Thus, in order to adapt the argument of Theorem 3.1 to periodic traveling waves, it remains only to show that the pressure trace on the bed furnishes Neumann data for h there. Unfortunately, because this setting lacks an upstream state, we will need some additional information.

More precisely, observe that by combining (3.1) with (3.2) for a general Bernoulli function β , we obtain the identity

$$(3.7) \quad h_p(q, p_0) = \left[2 \left(E|_{\psi=0} - \int_0^{p_0} \beta(p) dp + g\rho(p_0)d - P(q, p_0) \right) \right]^{-1/2}.$$

So long as we know $E|_{\psi=0}$, we can proceed as before. This leads to the following result.

Corollary 3.1. *Let $h \in C^{2,\alpha}(\overline{\mathcal{R}})$ be a periodic solution of the height equation (2.13) for a stable streamline density function $\rho \in C^\omega([p_0, 0])$, Bernoulli function $\beta \in C^\omega([0, |p_0|])$, and with wave speed c . Then $h \in C^\omega(\overline{\mathcal{R}})$. Moreover, h is uniquely determined by the trace of the pressure P on the bed and the value of E on the free surface.*

Remark 3.1. There are many situations in which it is possible to find $E|_{\psi=0}$ from other information about the flow. For instance, any one of the following is enough to determine $E|_{\psi=0}$:

- (i) the value of u at the crest and the height of the crest; or
- (ii) the value of u at the trough and the height of the trough; or
- (iii) the average value of the kinetic energy $\frac{\rho(0)}{2}|(u - c, v)|^2$ on the surface.

For (i), we just evaluate Bernoulli's law at the crest (where $v = 0$). Likewise, for (ii), we evaluate Bernoulli at the trough. For (iii), we take the mean value on the surface and use the fact that η has mean 0.

These determinability results can be easily extended to layer-wise smooth flows. Note that the interior interfaces are streamlines, hence they correspond to straight lines $\{p = p_i\}$ in the semi-Lagrangian variables. The first-order system (3.6) will hold in each strip \mathcal{R}_i . Beginning in the strip directly above the bed, we can argue as in Theorem 3.1 and Corollary 3.1 to show that $h \in C^\omega(\overline{\mathcal{R}_1})$ and is determined there uniquely by the prescribed data. h is continuous over the interface and indeed $q \mapsto h(q, \cdot)$ is real analytic. On the other hand, the jump condition (2.13c) allows us to determine $\partial_p h_2(\cdot, p_1)$. Thus we have a full set of Cauchy data for the first-order problem in \mathcal{R}_2 . Applying Cauchy-Kowalevski once more allows us to conclude $h \in C^\omega(\overline{\mathcal{R}_2})$. Iterating this procedure, we get the following result.

Corollary 3.2. *Let $h \in C^{0,\alpha}(\overline{\mathcal{R}})$ be a solitary wave solution of (2.13) with wave speed c such that $h_i \in C^{2,\alpha}(\overline{\mathcal{R}_i})$. Suppose that for each i , $\rho_i \in C^\omega([p_{i-1}, p_i])$ and $\beta_i \in C^\omega([|p_i|, |p_{i-1}|])$. Then h is real analytic in each layer \mathcal{R}_i , and it is uniquely determined by the trace of the pressure on the bed.*

3.2. Sobolev setting. In this section, we considerably strengthen the above results by relaxing the analyticity requirement to a more physically reasonable degree of smoothness. Our argument is built off of the following result on the strong unique continuation for a general elliptic equation set in the plane.

Theorem 3.2. [1, Theorem 1] *Given a bounded connected open set $\Omega \subset \mathbb{R}^2$. Let $u \in W^{1,2}(\Omega)$ be a weak solution to the elliptic equation*

$$Lu := (a^{ij}u_{x_i} + a^j u)_{x_j} + b^i u_{x_i} + cu = 0 \quad \text{in } \Omega,$$

where

- (i) L is uniformly elliptic in the sense that there exists a $K > 0$ such that

$$0 < K^{-1}|\zeta|^2 \leq a^{ij}(x)\zeta_i\zeta_j \leq K|\zeta|^2, \quad \text{for all } \zeta \in \mathbb{R}^2, \text{ and for a.e. } x \in \Omega; \text{ and}$$

- (ii) $a^{ij} \in L^\infty(\Omega)$, $a^i, b^i \in L^q(\Omega)$, $c \in L^{q/2}(\Omega)$ for some $q > 2$.

Then the operator L has the strong unique continuation property in the sense that if a solution u has a zero of infinite order at a point $x_0 \in \Omega$, i.e., there exists $\delta > 0$ such that for every integer $N \geq 0$

$$\int_{B_R(x_0)} |u|^2 dx \leq c_N r^N, \quad \text{for all } R < \delta,$$

then $u \equiv 0$ in Ω .

With this result in place, we can now state and prove our main theorem on determinability.

Theorem 3.3. *Let $r > 4$ and $h \in W^{3,r}(\mathcal{R})$ be a solution of (2.14) representing a solitary water wave with a stable streamline density stratification $\rho \in W^{2,r}([p_0, 0])$ and a Bernoulli function $\beta \in W^{1,r}([0, |p_0|])$, and such that the wave speed exceeds the horizontal fluid velocity. Then h is uniquely determined by the normal derivative h_p on the flat bed.*

Proof. Let $h, \tilde{h} \in W^{3,r}(\mathcal{R})$ be two solutions to (2.14) with identical Neumann data on the bed:

$$h_p(\cdot, p_0) = \tilde{h}_p(\cdot, p_0) \quad \text{on } \mathbb{R}.$$

The difference $u := h - \tilde{h}$ satisfies an elliptic equation (see also [43])

$$\mathcal{L}u := (a^{ij}u_{x_i})_{x_j} + b^i u_{x_i} + cu = 0 \quad \text{in } \mathcal{R},$$

where

$$\begin{aligned} a^{11} &:= 1 + h_q^2, & a^{12} &= a^{21} = -h_p h_q, & a^{22} &= h_p^2, \\ b^1 &:= \tilde{h}_{pp}(h_q + \tilde{h}_q) - 2h_p \tilde{h}_{pq} - h_p h_{pq} + h_q h_{pp}, \\ b^2 &:= \tilde{h}_{qq}(h_p + \tilde{h}_p) - 2\tilde{h}_q \tilde{h}_{pq} + \beta(-p)(h_p^2 + h_p \tilde{h}_p + \tilde{h}_p^2) - g\rho_p(\tilde{h} - d(\tilde{h}))(h_p^2 + h_p \tilde{h}_p + \tilde{h}_p^2) \\ &\quad - h_q h_{pq} + h_p h_{qq}, \\ c &:= -g\rho_p h_p^3, \end{aligned}$$

and

$$u = u_p = 0, \quad \text{on } \{p = p_0\}.$$

The above boundary condition guarantees that u can be extended to a $W^{1,r}$ function defined on the half-plane $\{p < 0\}$ that vanishes on $\{p < p_0\}$. Also, since $h, \tilde{h} \in W^{3,r}(\mathcal{R})$, from the trace theorem and fractional Sobolev embedding theorem (cf., e.g., [15, Theorem 8.2]) we know that

$$\nabla^2 h, \nabla^2 \tilde{h} \in W^{1/2,r}(\{p = 0\} \cup \{p = p_0\}) \subset C^{0, \frac{r-2}{2r}}(\{p = 0\} \cup \{p = p_0\}).$$

Similarly, we can make sense of ρ_p and β on $\{p = 0\}$ and $\{p = p_0\}$. This way, the functions h, \tilde{h}, ρ and β can be extended outside \mathcal{R} while maintaining their regularity and ensuring $h_p > 0$ in the strong sense. Therefore, from the divergence structure of the operator \mathcal{L} , we see that the extended u also satisfies $\mathcal{L}u = 0$. This, combined with Sobolev embedding, implies that the coefficients in \mathcal{L} satisfy the conditions of Theorem 3.2, and hence \mathcal{L} has the strong unique continuation property. Note that Theorem 3.2 is a local result. Thus we may apply it on any finite strip of the fluid domain to show that the restriction of u there vanishes identically. It follows that $u \equiv 0$ on the entire strip, completing the proof of theorem. \square

As discussed in the previous subsection, $h_p(\cdot, p_0)$ is determined by $P(\cdot, p_0)$ through (3.4). Thus Theorem 3.1 can be reinterpreted as a statement about unique determinability of h from the pressure trace.

Theorem 3.4. *Let $r > 2$ and $h \in W^{3,r}(\mathcal{R})$ be a solution of (2.14) representing a solitary water wave with a stable streamline density stratification $\rho \in W^{2,r}([p_0, 0])$, a Bernoulli function $\beta \in W^{1,r}([p_0, 0])$. Fix the wave speed c , and the downstream/upstream horizontal velocity on the*

surface. Then h is uniquely determined by the trace of the pressure P on the bed. In particular, the pressure function on the flat bed uniquely determines the free surface profile.

Remark 3.2. The same argument applies to the regime considered by Henry [19], namely constant density flow with a general vorticity distribution. In that case, the height equation is

$$(3.8) \quad \begin{cases} (1 + h_q^2)h_{pp} + h_{qq}h_p^2 - 2h_qh_ph_{pq} = -h_p^3\gamma(-p), & p_0 < p < 0, \\ 1 + h_q^2 + h_p^2(2gh - Q) = 0, & p = 0, \\ h = 0, & p = p_0, \end{cases}$$

where γ is the vorticity function. Clearly γ plays a nearly identical role as β does in (2.14), and so the arguments above go through with only minor alteration.

4. RECONSTRUCTION FOR LAYER-WISE CONSTANT DENSITY IRROTATIONAL WAVES

Up to this point, we have proved that, in a number of physical regimes, the pressure trace on the bed uniquely determines a traveling wave in a stratified water. The goal of the present section is to develop an explicit iteration scheme that reconstructs the internal interfaces of a layer-wise constant density and irrotational solitary wave given the pressure data.

4.1. Surface reconstruction from pressure. In Section 2.4, we introduced the hodograph transform and the reformulation of the height equation in (t, s) -coordinates. Recall that the density is constant within each layer and we denote the height function as $H = H(t, s)$ when working in the conformal variables. Reading off the Bernoulli law in the bottom layer from far-field data yields

$$(4.1) \quad P_{\text{bot}} + \frac{c^2}{2[(\partial_t H_1)^2 + (\partial_s H_1)^2]} - g\varrho_1 d = P_{\text{atm}} + \int_{p_0}^0 g\rho(p) dp + \frac{\varrho_1 c^2}{2} - g\varrho_1 d,$$

where the second term on the right-hand side comes from inverting the streamline coordinates from the Eulerian one at infinity. The boundary condition on H_1 , (2.19), and the fact that $H_s > 0$ then give a formula for $\partial_s H_1(\cdot, s_0)$

$$(4.2) \quad \partial_s H_1(t, s_0) = \frac{c}{\sqrt{\varrho_1} \sqrt{c^2 - 2\mathfrak{p}(t)}},$$

where

$$(4.3) \quad \mathfrak{p}(t) := \frac{P_{\text{bot}}(x_0(t)) - P_{\text{atm}}}{\varrho_1} - \frac{1}{\varrho_1} \int_{p_0}^0 g\rho(p) dp.$$

Recall that x_0 is the inverse of $x \mapsto \varphi_1(x, -d)$.

The surface reconstruction problem can then be stated as follows: From the form of the wave at infinity as described by the data \hat{H} , d , ρ , and c , and the pressure trace on the bed as described by \mathfrak{p} , find a parameterization for each of the internal interfaces as well as the air–sea interface.

Remark 4.1. \mathfrak{p} is a more convenient quantity to work with mathematically, but will not be directly observable in most physical scenarios. However, it can be determined uniquely from P_{bot} , the trace of the pressure on the bed in the original variables. This follows from several

observations. First, note that the vertical velocity vanishes on the bed, and thus from Bernoulli's law we find that:

$$\varphi_x(x, -d) = - \left[P_{\text{atm}} - P_{\text{bot}}(x) + \int_{p_0}^0 g\rho(p) dp + \frac{\varrho_1 c^2}{2} \right]^{1/2}.$$

Here we are using the fact that the absence of stagnation implies $\varphi_x(0, -d) = \sqrt{\rho_1}(u - c) < 0$, which selects the negative branch of the square root. Thus x_0 is characterized as the solution to the ODE initial value problem

$$\begin{cases} \dot{x}_0(t) = \frac{1}{\varphi_x(x_0(t))} = - \left[P_{\text{atm}} - P_{\text{bot}}(x_0(t)) + \int_{p_0}^0 g\rho(p) dp + \frac{\varrho_1 c^2}{2} \right]^{-1/2} \\ x_0(0) = 0. \end{cases}$$

The condition $x_0(0) = 0$ follows from our choice to take $\varphi(0, -d) = 0$. Once x_0 is known, \mathbf{p} is determined uniquely by (4.3).

It is important to note that one cannot hope to choose \mathbf{p} arbitrarily and have it correspond to the pressure trace of an actual traveling wave. We will require a posteriori information about H ; that is, we need to make use of the fact that it is known that we are observing a solution, and that solutions of the system have a certain regularity and decay at infinity. Otherwise, the ill-posedness of the problem cannot be avoided.

Fix an $\alpha \in (0, 1)$. We will assume that the wave to be reconstructed lies in the set

$$(4.4) \quad \mathcal{H} = \mathcal{H}(\rho, \dot{H}, c, d) := \{H \in C^{0,\alpha}(\overline{\mathcal{T}}) : H - \dot{H}, H_t \text{ exponentially localized in } t, \\ H \text{ a solution of (2.21)}\}.$$

The existence theory of Turner [40, 41] shows that \mathcal{H} is non-empty if the wave speed is supercritical (cf. Section 5). In [4], moreover, Amick and Turner prove that it includes waves of large amplitude in the case of a two-layer system.

With this degree of regularity, our procedure holds in the strong sense; with less decay at infinity, we obtain only a distributional representation for the wave. Of course, since $H \in \mathcal{H}$ is harmonic in each layer, it is $C^\infty(\overline{\mathcal{T}}_i)$ for $i = 1, \dots, N$. In fact, the structure of the equation and regularity theory for transmission boundary conditions (cf., e.g., [28, Chapter 16]) imply that $\partial_t H \in C^{0,\alpha}(\overline{\mathcal{T}})$.

4.2. Jump condition in conformal coordinate system. One of the main technical difficulties for multi-layered flows is that the conformal coordinates behave badly on the image of the interfaces. Suppose that $F \in C^0(\overline{\Omega})$ and let $f := F \circ \mathcal{K}^{-1}$. Then, in general, $f \notin C^0(\overline{\mathcal{T}})$ since

$$f_{i+1}(t, s_i) \neq f_i(t, s_i).$$

In other words, while each \mathcal{K}_i is a conformal mapping, \mathcal{K} itself is not a homeomorphism. Of course, this stems from the fact that φ may have a jump discontinuity over the interface. Recall that we chose in (2.16) to normalize φ so that $\varphi^{-1}(\{0\})$ is a connected set. In terms of the hodograph transform, this becomes

$$(\mathcal{K}_{i+1})^{-1}(0, s_i) = (\mathcal{K}_i)^{-1}(0, s_i), \quad \text{for all } i = 1, \dots, N-1,$$

which ensures that, at the very least, functions like f above are continuous along the s -axis.

Given this issue, it is necessary to examine the jump conditions on the internal interfaces very carefully. Consider the situation on the i -th interior interface $\{s = s_i\}$, for some $1 \leq i \leq N - 1$. Let

$$(4.5) \quad X_{i+} := x_{i+1}(\cdot, s_i), \quad X_{i-} := x_i(\cdot, s_i),$$

where $x_i(t, s)$ is the restriction of the Eulerian x -coordinate on \mathcal{T}_i . Likewise define

$$\Phi_{i+}(x) := \varphi_{i+1}(x, \eta_i(x)), \quad \Phi_{i-}(x) := \varphi_i(x, \eta_i(x)).$$

This simply says that $X_{i\pm}$ and $\Phi_{i\pm}$ are inverse to each other, i.e.,

$$(4.6) \quad X_{i\pm} = X_{i\mp}(\Phi_{i\mp}(X_{i\pm})), \quad t = \Phi_{i\pm}(X_{i\pm}(t)).$$

The relation (4.6) allows us to translate from the parameterization of the interface by the coordinates in \mathcal{T}_{i+1} , to that given by the coordinates in \mathcal{T}_i : define

$$(4.7) \quad \tau_{i\pm}(t) := \Phi_{i\pm}(X_{i\mp}(t)),$$

then

$$(4.8) \quad (\mathcal{K}_{i+1})^{-1}(t, s_i) = (\mathcal{K}_i)^{-1}(\tau_{i-}(t), s_i), \quad (\mathcal{K}_{i+1})^{-1}(\tau_{i+}(t), s_i) = (\mathcal{K}_i)^{-1}(t, s_i).$$

We also mention that (2.16) implies

$$(4.9) \quad X_{i+}(0) = X_{i-}(0) \quad \text{and} \quad \tau_{i-}(0) = 0.$$

Suppose that f is a function defined on $\overline{\mathcal{T}_i} \cup \overline{\mathcal{T}_{i+1}}$ and let $F = f \circ \mathcal{K}$ be its expression in physical variables. Then $f_{i+1}(t, s_i)$ corresponds to the value of F_{i+1} at the Eulerian point $(x, \eta_i(x))$ where $\Phi_{i+}(x) = t$. Chasing definitions, we see that this is the same point at which $\Phi_{i-}(x) = \tau_{i-}(t)$. Therefore, the jump in F over the i -th interface corresponds to the following difference in the conformal variables:

$$(4.10) \quad \llbracket f \rrbracket_i := f_{i+1}(\cdot, s_i) - f_i(\tau_{i-}(\cdot), s_i).$$

Hence the jump condition (2.12c) becomes

$$(4.11) \quad \left\| \frac{c^2}{H_t^2 + H_s^2} \right\|_i + 2g \left\| \varrho(H - \dot{H}) \right\|_i = \llbracket \varrho \rrbracket_i c^2, \quad \text{on } \{s = s_i\}, \quad i = 1, \dots, N - 1.$$

4.3. Solution of the Cauchy problem. Our procedure begins with the pressure data on the ocean bed, then reconstructs the flow in each layer moving from the lowest layer upward. At each step, this involves solving a non-standard elliptic problem: H is harmonic in each strip \mathcal{T}_i , and we will have both the Dirichlet and Neumann data on the lower boundary. This is essentially the Cauchy problem for Laplace's equation (cf. Section 3), which is generally ill-posed. In the present case, however, we are not concerned with existence theory — we know the wave exists because we have measured it — and the asymptotic decay properties (2.22) will allow us to make sense of the solutions as functions. The ill-posedness issue reemerges if the pressure data is noisy, but we forestall that discussion until Section 5.4.

The process of solving the Cauchy problem in each strip can be abstracted using nonlocal operators. In fact, these operators are nothing but Fourier multipliers whose symbols can be

found via elementary separation of variables. Explicitly, for each $i = 1, \dots, N$, we define an operator \mathcal{C}_i by

$$(4.12) \quad \mathcal{C}_i(\phi_1, \phi_2) := (H|_{s=s_i}, H_s|_{s=s_i}),$$

where H is the unique solution of the Cauchy problem

$$(4.13) \quad \begin{cases} \Delta H = 0, & \text{in } \mathcal{T}_i \\ H = \phi_1, \quad H_s = \phi_2, & \text{on } \{s = s_{i-1}\}. \end{cases}$$

That is, \mathcal{C}_i takes the Cauchy data on the lower boundary of the $(i-1)$ -st layer $\{s = s_{i-1}\}$, solves the PDE, and then evaluates the Cauchy data for that solution on the upper boundary $\{s = s_i\}$. For now, we do not specify precisely the domain and codomain of \mathcal{C}_i , but rather work formally.

Taking the Fourier transform of (4.13) in the t -direction, denoting by ξ the Fourier variable, we see immediately that $\widehat{H} := \mathcal{F}H$ must take the form

$$(4.14) \quad \widehat{H}(\xi, s) = \widehat{\phi}_1(\xi) \cosh[(s - s_{i-1})\xi] + \frac{\widehat{\phi}_2(\xi)}{\xi} \sinh[(s - s_{i-1})\xi].$$

Hence

$$(4.15) \quad \mathcal{F}[\mathcal{C}_i(\phi_1, \phi_2)](\xi) = \begin{pmatrix} \widehat{\phi}_1(\xi) \cosh[(s_i - s_{i-1})\xi] + \frac{\widehat{\phi}_2(\xi)}{\xi} \sinh[(s_i - s_{i-1})\xi] \\ \xi \widehat{\phi}_1(\xi) \sinh[(s_i - s_{i-1})\xi] + \widehat{\phi}_2(\xi) \cosh[(s_i - s_{i-1})\xi] \end{pmatrix}^T.$$

4.4. Reconstruction of the first internal interface. As a model computation, we derive a reconstruction formula for the first interface $\{y = \eta_1(x)\}$ given $P_{\text{bot}}(\cdot, s_0)$. From (4.2) we recover the normal derivative of H_1 on the bed

$$(4.16) \quad \partial_s H_1(t, s_0) = \frac{c}{\sqrt{\varrho_1} \sqrt{c^2 - 2\mathbf{p}(t)}}.$$

Note that both sides above are bounded functions of t and hence can be regarded as tempered distributions. Performing a Fourier transform in t yields

$$\widehat{\partial_s H_1}(\xi, s_0) = \mathcal{F} \left\{ \frac{c}{\sqrt{\varrho_1} \sqrt{c^2 - 2\mathbf{p}}} \right\}(\xi)$$

in the sense of distributions. Applying (4.15), we find that

$$(4.17) \quad \widehat{H}_1(\xi, s) = \frac{\sinh[(s - s_0)\xi]}{\xi} \mathcal{F} \left\{ \frac{c}{\sqrt{\varrho_1} \sqrt{c^2 - 2\mathbf{p}}} \right\}(\xi), \quad s \in [s_0, s_1],$$

$$(4.18) \quad \widehat{\partial_s H_1}(\xi, s_1) = \cosh[(s_1 - s_0)\xi] \mathcal{F} \left\{ \frac{c}{\sqrt{\varrho_1} \sqrt{c^2 - 2\mathbf{p}}} \right\}(\xi).$$

From the definitions of x_i, H_i , and \mathcal{K} , we know that the inverse of $\mathcal{K}|_{\overline{\mathcal{T}_1}}$ is given by

$$x = x_1(t, s), \quad y = H_1(t, s) - d.$$

This together with (4.5) and (2.17) yields

$$(4.19) \quad (x, \eta_1(x)) = (x_1(t, s_1), H_1(t, s_1) - d) = (X_{1-}(t), Y_1(t)).$$

Note that there is no reason to add a \pm to the subscript for Y_1 because $s \mapsto y(\cdot, s)$, unlike $s \mapsto x(\cdot, s)$, is continuous.

From (2.19) we have the identity $\partial_s H_1 = \partial_t x_1$. Therefore, (4.17) and (4.18) give

$$\begin{aligned}\mathcal{F}\{Y_1\}(\xi) &= \frac{\sinh[(s_1 - s_0)\xi]}{\xi} \mathcal{F}\left\{\frac{c}{\sqrt{\varrho_1}\sqrt{c^2 - 2\mathbf{p}}}\right\}(\xi) - 2\pi d\delta(\xi), \\ \mathcal{F}\{\dot{X}_{1-}\}(\xi) &= \cosh[(s_1 - s_0)\xi] \mathcal{F}\left\{\frac{c}{\sqrt{\varrho_1}\sqrt{c^2 - 2\mathbf{p}}}\right\}(\xi),\end{aligned}$$

since $\mathcal{F}\{1\}(\xi) = 2\pi\delta(\xi)$. Using some elementary identities, we can simplify this further to obtain

$$(4.20) \quad \mathcal{F}\left\{Y_1 - \left(\frac{s_1 - s_0}{\sqrt{\varrho_1}} - d\right)\right\}(\xi) = \frac{\sinh[(s_1 - s_0)\xi]}{\xi} \mathcal{F}\left\{\frac{1}{\sqrt{\varrho_1}} \left(\frac{c}{\sqrt{c^2 - 2\mathbf{p}}} - 1\right)\right\}(\xi),$$

$$(4.21) \quad \mathcal{F}\left\{\dot{X}_{1-} - \frac{1}{\sqrt{\varrho_1}}\right\}(\xi) = \cosh[(s_1 - s_0)\xi] \mathcal{F}\left\{\frac{1}{\sqrt{\varrho_1}} \left(\frac{c}{\sqrt{c^2 - 2\mathbf{p}}} - 1\right)\right\}(\xi).$$

Both (4.20) and (4.21) hold in the sense of (tempered) distributions. Ideally, we would like them to hold in the strong sense, and to show this we use the idea of Constantin [11]. From (2.19) and (4.2) we know that

$$(4.22) \quad [c - u(x, -d)]^2 = c^2 - 2\mathbf{p}(t).$$

On the other hand, we are working with solutions in the set \mathcal{H} , which means that $u(x, -d)$, $\nabla u(x, -d)$ decay exponentially fast as $|x| \rightarrow \infty$. So, by combining (4.22) and (2.20), we may conclude that \mathbf{p} and its derivative are square-integrable. In particular, $\mathbf{p}(t) \rightarrow 0$ as $|t| \rightarrow \infty$. Moreover, writing

$$\frac{c}{\sqrt{c^2 - 2\mathbf{p}(t)}} - 1 = \frac{2\mathbf{p}}{\sqrt{c^2 - 2\mathbf{p}(t)}(c + \sqrt{c^2 - 2\mathbf{p}(t)})},$$

we see that the left-hand side is likewise in L^2 .

Observe from (2.19) and (4.5) that $\dot{X}_{1-} = \partial_s H_1(\cdot, s_1)$, and therefore the terms on the left-hand side of (4.20) and (4.21) are Fourier transforms of square integrable functions. This guarantees that (4.20) and (4.21) are indeed equalities for functions. Putting together (4.19)–(4.21), we obtain a parameterization of the first internal interface

$$(4.23) \quad X_{1-}(t) = \frac{t}{\sqrt{\varrho_1}} + \int_{-\infty}^t \mathcal{F}^{-1}\left\{\cosh[(s_1 - s_0)\xi] \mathcal{F}\left[\frac{1}{\sqrt{\varrho_1}} \left(\frac{c}{\sqrt{c^2 - 2\mathbf{p}}} - 1\right)\right](\xi)\right\}(t') dt',$$

$$(4.24) \quad Y_1(t) = \mathcal{F}^{-1}\left\{\frac{\sinh[(s_1 - s_0)\xi]}{\xi} \mathcal{F}\left[\frac{1}{\sqrt{\varrho_1}} \left(\frac{c}{\sqrt{c^2 - 2\mathbf{p}}} - 1\right)\right](\xi)\right\}(t) + \frac{s_1 - s_0}{\sqrt{\varrho_1}} - d.$$

4.5. Reparameterization. Now suppose we have obtained the parametrization of the i -th interface $(X_{i-}(t), Y_i(t))$. In order to iterate the procedure given above, we will need to compute the function τ_{i-} that allows us to deal with the singularity in the coordinates on the interface. It is

simplest to do this working in the semi-Lagrangian variables. Recall that the jump condition on $\{s = s_i\}$ (which is also $\{p = p_i\}$) is expressed as follows

$$-\frac{1 + (\partial_q h_{i+1})^2}{2(\partial_p h_{i+1})^2} + \frac{1 + (\partial_q h_i)^2}{2(\partial_p h_i)^2} - g(\varrho_{i+1} h_{i+1} - \varrho_i h_i) + \frac{Q_i}{2} = 0 \quad \text{on } \{p = p_i\},$$

where

$$Q_i := 2[(E_{i+1} - E_i) + g(\varrho_{i+1} - \varrho_i)d].$$

Using the fact that h and h_q are both continuous across the interface we can solve for $\partial_p h_{i+1}$

$$(4.25) \quad \partial_p h_{i+1} = \sqrt{1 + (\partial_q h_i)^2} \left[Q_i - 2g(\varrho_{i+1} - \varrho_i)h_i + \frac{1 + (\partial_q h_i)^2}{(\partial_p h_i)^2} \right]^{-1/2} \quad \text{on } \{p = p_i\}.$$

Recall that we have the change of variable relations

$$\frac{1}{h_p} = -\psi_y = \varphi_x, \quad \frac{h_q}{h_p} = \psi_x = \varphi_y.$$

Thus, from (4.25) we can recover the full gradient $\nabla \varphi$ at each point on the i -th interface:

$$\begin{aligned} \partial_x \varphi_{i+1}(x, \eta_i(x)) &= \frac{1}{\partial_p h_{i+1}(x, p_i)} = [1 + (\partial_q h_i)^2]^{-1/2} \left[Q_i - 2g(\varrho_{i+1} - \varrho_i)h_i + \frac{1 + (\partial_q h_i)^2}{(\partial_p h_i)^2} \right]^{1/2} \Big|_{(x, p_i)}, \\ \partial_y \varphi_{i+1}(x, \eta_i(x)) &= \frac{\partial_q h_{i+1}(x, p_i)}{\partial_p h_{i+1}(x, p_i)} = \frac{\partial_q h_i}{\sqrt{1 + (\partial_q h_i)^2}} \left[Q_i - 2g(\varrho_{i+1} - \varrho_i)h_i + \frac{1 + (\partial_q h_i)^2}{(\partial_p h_i)^2} \right]^{1/2} \Big|_{(x, p_i)}. \end{aligned}$$

Let us now compute τ_{i-} . By differentiating (4.7) and using the identity

$$\Phi_{i-}(X_{i+}) = \varphi_i(X_{i+}, \eta_i(X_{i+})),$$

we obtain

$$\begin{aligned} \dot{\tau}_{i-}(t) &= \dot{X}_{i+}(1, \eta'_i) \cdot \nabla \varphi_i(X_{i+}, \eta_i) \\ (4.26) \quad &= \dot{X}_{i+} \frac{1 + (\partial_q h_i)^2}{\partial_p h_i} \Big|_{(X_{i+}, p_i)} = \frac{c \dot{X}_{i+}(t)}{(\partial_s H_i(X_{i+}(t), s_i))^2}. \end{aligned}$$

Here a dot denotes a derivative with respect to t . Integrating, and rewriting in terms of the parameterization from the lower layer, this becomes

$$\tau_{i-}(t) = \tau_{i-}(0) + \int_{X_{i+}(0)}^{X_{i+}(t)} \frac{c}{(\partial_s H_i)(\tau, s_i)^2} d\tau = \int_{X_{i+}(0)}^{X_{i+}(t)} \frac{c}{(\dot{X}_{i-}(t'))^2} dt',$$

where we have used the fact that $\tau_{i-}(0) = 0$ by (4.9).

Likewise, an ODE characterizing X_{i+} can be derived by differentiating the second identity in (4.6), giving

$$\begin{aligned} 1 &= \Phi'_{i+}(X_{i+}) \dot{X}_{i+} = (1, \eta'_i) \cdot \nabla \varphi_{i+1}(X_{i+}, \eta_i) \dot{X}_{i+} \\ &= \dot{X}_{i+} \sqrt{1 + (\partial_q h_i)^2} \left[Q_i - 2g(\varrho_{i+1} - \varrho_i)h_i + \frac{1 + (\partial_q h_i)^2}{(\partial_p h_i)^2} \right]^{1/2} \Big|_{(X_{i+}, p_i)}. \end{aligned}$$

Again, initial conditions for this ODE are fixed by (4.9), and thus X_{i+} can be defined as the unique solution of

$$\begin{cases} \dot{X}_{i+}(t) = \frac{\dot{X}_{i-}}{\sqrt{\dot{X}_{i-}^2 + \dot{Y}_i^2}} \left[Q_i - 2g(\varrho_{i+1} - \varrho_i)(Y_i + d) + \frac{c^2}{\varrho_i(Y_i^2 + (\dot{X}_{i-})^2)} \right]^{-1/2} \Big|_{(X_{i+}, s_i)}, \\ X_{i+}(0) = X_{i-}(0). \end{cases}$$

In this way, we can determine τ_{i-} and X_{i+} from $(X_{i-}, Y_i, \dot{X}_{i-})$.

4.6. Iteration. In the previous subsections, we saw how to recover the first internal interface, and how to handle the jump conditions there. The next step is to iterate this process so that we can reconstruct the entire wave. This is done in two stages: first, we obtain formulas that hold in the distributional sense. Second, we use the fact that we are working with $H \in \mathcal{H}$ to deduce that they hold in the strong sense as well.

4.6.1. Iteration in the sense of distributions. For each $i \geq 1$, we define the trace operator Γ_i with domain \mathcal{H} by

$$\Gamma_i : H \mapsto (H_i(\cdot, s_i), (\partial_s H_i)(\cdot, s_i), (\partial_t H_i)(\cdot, s_i)) \in \mathcal{H}_i,$$

where

$$\Gamma_i(\mathcal{H}) =: \mathcal{H}_i = \mathcal{H}_i^{(1)} \times \mathcal{H}_i^{(2)} \times \mathcal{H}_i^{(3)}.$$

Simply put, Γ_i evaluates the traces of the solution H and its first derivatives in the region \mathcal{T}_i on the interface $\{s = s_i\}$. The data we are given to begin the reconstruction process is precisely

$$\mathcal{H}_0 := \left(0, \frac{c}{\sqrt{\varrho_1} \sqrt{c^2 - 2\mathbf{p}(t)}}, 0 \right).$$

The iteration procedure is as follows. In the $(i+1)$ -st fluid domain \mathcal{T}_{i+1} , we are solving a Laplace equation with Dirichlet-Neumann data on the bottom boundary $\{s = s_i\}$ obtained from previous step: Take the data $(H_i(\cdot, s_i), (\partial_s H_i)(\cdot, s_i), (\partial_t H_i)(\cdot, s_i))$ from the solution H_i in \mathcal{T}_i . In order to recast the hodograph transformation in \mathcal{T}_{i+1} we need to access to τ_{i-} computed in the previous subsection. From (4.8), the continuity of H and $\partial_t H$ across the interface $\{s = s_i\}$ can be expressed as

$$H_{i+1}(t, s_i) = H_i(\tau_{i-}(t), s_i), \quad \partial_t H_{i+1}(t, s_i) = \partial_t H_i(\tau_{i-}(t), s_i).$$

This allows us to compute $\partial_s H_{i+1}(\cdot, s_i)$ from (4.11):

$$(4.27) \quad \begin{aligned} \partial_s H_{i+1}(\cdot, s_i)^2 = & \left[\frac{1}{\partial_t H_i(\tau_{i-}(\cdot), s_i)^2 + \partial_s H_i(\tau_{i-}(\cdot), s_i)^2} \right. \\ & \left. + \frac{c^2 - 2g \left[H_i(\tau_{i-}(\cdot), s_i) - \hat{H}(s_i) \right]}{c^2} (\varrho_{i+1} - \varrho_i) \right]^{-1} - \partial_t H_i(\cdot, s_i)^2. \end{aligned}$$

We have therefore found a complete set of Cauchy data $(H_{i+1}, \partial_s H_{i+1}, \partial_t H_{i+1})$ for the Laplace equation (4.13) in the next layer \mathcal{T}_{i+1} . Solving this, gives $(H_{i+1}(\cdot, s_{i+1}), (\partial_s H_{i+1})(\cdot, s_{i+1}), (\partial_t H_{i+1})(\cdot, s_{i+1}))$. In total, this procedure can be written abstractly as a mapping

$$\mathcal{I}_{i+1} := \mathcal{H}_i \rightarrow \mathcal{H}_{i+1}, \quad i = 1, \dots, N-1.$$

The nonlocal and nonlinear expression for \mathcal{I}_{i+1} is given by:

$$(4.28) \quad \begin{aligned} \mathcal{I}_{i+1}^{(1)}(u, v, w) &= \mathcal{F}^{-1} \left\{ \cosh[(s_{i+1} - s_i)\xi] \widehat{u \circ \tau_{i-}}(\xi) + \frac{\sinh[(s_{i+1} - s_i)\xi]}{\xi} \widehat{f_{i+1}}(u, v, w)(\xi) \right\} \\ \mathcal{I}_{i+1}^{(2)}(u, v, w) &= \mathcal{F}^{-1} \left\{ \xi \sinh[(s_{i+1} - s_i)\xi] \widehat{u \circ \tau_{i-}}(\xi) + \cosh[(s_{i+1} - s_i)\xi] \widehat{f_{i+1}}(u, v, w)(\xi) \right\} \\ \mathcal{I}_{i+1}^{(3)}(u, v, w) &= \partial_t(u \circ \tau_{i-}), \end{aligned}$$

where

$$(4.29) \quad f_{i+1}(u, v, w)(\cdot) := \left[\frac{1}{\frac{1}{v(\tau_{i-}(\cdot))^2 + w(\tau_{i-}(\cdot))^2} - \frac{c^2 + 2g[u(\tau_{i-}(\cdot)) - \mathring{H}(s_i)]}{c^2}} - w^2 \right]^{1/2},$$

and the Fourier transforms are taken in the sense of distributions.

4.6.2. Iteration in the sense of functions. The above scheme can be improved so that the equalities hold in the sense of functions. To do this, we utilize the exponential asymptotics (2.22) to conclude that

$$(4.30) \quad \begin{aligned} H_i(\tau_{i-}(t), s_i) &= H_{i+1}(t, s_i) \rightarrow \mathring{H}(s_i), \quad H_{i+1}(t, s_{i+1}) \rightarrow \mathring{H}(s_{i+1}), \\ \partial_s H_{i+1}(t, s) &\rightarrow \frac{1}{\sqrt{\varrho_{i+1}}}, \quad \text{for } s \in [s_i, s_{i+1}], \end{aligned} \quad \text{as } |t| \rightarrow \infty,$$

with the convergence being exponentially fast. Thus, the differences between the functions and their corresponding asymptotics are square integrable. On the other hand, plugging (4.30) into the first equation of (4.28), we find that

$$\begin{aligned} \mathcal{F} \left\{ H_{i+1}(\cdot, s_{i+1}) - \mathring{H}(s_i) - \frac{s_{i+1} - s_i}{\sqrt{\varrho_{i+1}}} \right\}(\xi) &= \cosh[(s_{i+1} - s_i)\xi] \mathcal{F} \left\{ H_i(\tau_{i-}(\cdot), s_i) - \mathring{H}(s_i) \right\}(\xi) \\ &\quad + \frac{\sinh[(s_{i+1} - s_i)\xi]}{\xi} \mathcal{F} \left\{ \partial_s H_{i+1}(\cdot, s_i) - \frac{1}{\sqrt{\varrho_{i+1}}} \right\}(\xi), \end{aligned}$$

where $\partial_s H_{i+1}(\cdot, s_i)$ is determined from (4.27). Further simplifying the left-hand side above by using the definition of \mathring{H} in (2.23), we see that

$$(4.31) \quad \begin{aligned} \mathcal{F} \left\{ H_{i+1}(t, s_{i+1}) - \mathring{H}(s_{i+1}) \right\}(\xi) &= \cosh[(s_{i+1} - s_i)\xi] \mathcal{F} \left\{ H_i(\tau_{i-}(\cdot), s_i) - \mathring{H}(s_i) \right\}(\xi) \\ &\quad + \frac{\sinh[(s_{i+1} - s_i)\xi]}{\xi} \mathcal{F} \left\{ \partial_s H_{i+1}(\cdot, s_i) - \frac{1}{\sqrt{\varrho_{i+1}}} \right\}(\xi), \end{aligned}$$

in the strong sense.

Now, to transform back to the original variables, consider \mathcal{K}_{i+1}^{-1} :

$$x = x_{i+1}(t, s), \quad y = H_{i+1}(t, s) - d.$$

From the relation $\partial_t x_{i+1} = \partial_s H_{i+1}$ and (4.15), we know that

$$\begin{aligned} \mathcal{F}\{\partial_t x_{i+1}(\cdot, s_{i+1})\}(\xi) &= \xi \sinh[(s_{i+1} - s_i)\xi] \mathcal{F}\{H_i(\tau_{i-}(\cdot), s_i)\}(\xi) \\ &\quad + \cosh[(s_{i+1} - s_i)\xi] \mathcal{F}\{\partial_s H_{i+1}(\cdot, s_i)\}(\xi). \end{aligned}$$

Similarly, using the asymptotics (4.30),

$$\partial_t x_{i+1}(t, s_{i+1}) = \partial_s H_{i+1}(t, s_{i+1}) \rightarrow \frac{1}{\sqrt{\varrho_{i+1}}} \quad \text{exponentially as } |t| \rightarrow \infty,$$

and the identity $\xi \sinh(a\xi) \mathcal{F}\{1\}(\xi) = \xi \sinh(a\xi) 2\pi\delta(\xi) = 0$, we ultimately conclude that

$$\begin{aligned} (4.32) \quad \mathcal{F}\left\{\partial_t x_{i+1}(\cdot, s_{i+1}) - \frac{1}{\sqrt{\varrho_{i+1}}}\right\}(\xi) &= \xi \sinh[(s_{i+1} - s_i)\xi] \mathcal{F}\left\{H_i(\tau_{i-}(\cdot), s_i) - \mathring{H}(s_i)\right\}(\xi) \\ &\quad + \cosh[(s_{i+1} - s_i)\xi] \mathcal{F}\left\{\partial_s H_{i+1}(\cdot, s_i) - \frac{1}{\sqrt{\varrho_{i+1}}}\right\}(\xi). \end{aligned}$$

Note that $X_{(i+1)-}(t) = x_{i+1}(t, s_{i+1})$ and $Y_{i+1}(t) = H_{i+1}(t, s_{i+1})$. Hence the $(i+1)$ -st interface can be parametrized by

$$\begin{aligned} (4.33) \quad X_{(i+1)-}(t) &= \frac{t}{\sqrt{\varrho_{i+1}}} + \int_{-\infty}^t \mathcal{F}^{-1}\left\{\xi \sinh[(s_{i+1} - s_i)\xi] \mathcal{F}\left[H_i(\tau_{i-}(t), s_i) - \mathring{H}(s_{i+1})\right](\xi) \right. \\ &\quad \left. + \cosh[(s_{i+1} - s_i)\xi] \mathcal{F}\left[\partial_s H_{i+1}(t, s_i) - \frac{1}{\sqrt{\varrho_{i+1}}}\right](\xi)\right\}(t') dt' \\ (4.34) \quad Y_{i+1}(t) &= \mathcal{F}^{-1}\left\{\cosh[(s_{i+1} - s_i)\xi] \mathcal{F}\left[H_i(\tau_{i-}(t), s_i) - \mathring{H}(s_{i+1})\right](\xi) \right. \\ &\quad \left. + \frac{\sinh[(s_{i+1} - s_i)\xi]}{\xi} \mathcal{F}\left[\partial_s H_{i+1}(t, s_i) - \frac{1}{\sqrt{\varrho_{i+1}}}\right](\xi)\right\}(t) + \mathring{H}(s_{i+1}) - d. \end{aligned}$$

An argument like that given at the end of Section 4.4 shows that these are equalities in the sense of functions.

In summary, to iterate in the regular function setting, we replace the trace operator Γ_i in the previous subsection by another one which evaluates the difference between the trace and its asymptotics

$$\tilde{\Gamma}_i(\mathcal{H}) = \left(H_i(\cdot, s_i) - \mathring{H}(s_i), (\partial_s H_i)(\cdot, s_i) - \frac{1}{\sqrt{\varrho_i}}, (\partial_t H_i)(\cdot, s_i)\right) =: \tilde{\mathcal{H}}_i, \quad i = 1, \dots, N.$$

Likewise, we normalized the data on the bed so that it has a Fourier transform defined in the strong sense:

$$\tilde{\mathcal{H}}_0 := \left(0, \frac{c}{\sqrt{\varrho_1} \sqrt{c^2 - 2\mathbf{p}(t)}} - \frac{1}{\sqrt{\varrho_1}}, 0\right).$$

A corresponding reconstruction map for the height function $\tilde{\mathcal{I}}_{i+1} : \tilde{\mathcal{H}}_i \rightarrow \tilde{\mathcal{H}}_{i+1}$, $i = 1, \dots, N-1$ can thus be constructed as follows.

$$(4.35) \quad \begin{aligned} \tilde{\mathcal{I}}_{i+1}^{(1)} &= \mathcal{F}^{-1} \{ \text{the right-hand side of (4.31)} \}, \text{ where } \partial_s H_{i+1}(\cdot, s_i) \text{ is given in (4.27),} \\ \tilde{\mathcal{I}}_{i+1}^{(2)} &= \partial_t x_{i+1}(t, s_{i+1}) - \frac{1}{\sqrt{\varrho_{i+1}}} = \mathcal{F}^{-1} \{ \text{the right-hand side of (4.32)} \}, \\ \tilde{\mathcal{I}}_{i+1}^{(3)} &= \partial_t \tilde{\mathcal{I}}_{i+1}^{(1)}, \end{aligned}$$

which the corresponding iteration scheme for the parametrized interface reconstruction is

$$(4.36) \quad \begin{cases} X_{(i+1)-}(t) = \frac{t}{\sqrt{\varrho_{i+1}}} + \int_{-\infty}^t \tilde{\mathcal{I}}_{i+1}^{(2)}(t') dt', \\ Y_{i+1}(t) = \tilde{\mathcal{I}}_{i+1}^{(1)}(t) + \mathring{H}(s_{i+1}) - d, \end{cases} \quad i = 1, \dots, N-1.$$

Composition then gives the $(i+1)$ -st layer directly from the data on the ocean floor (which can be determined from the bottom pressure readings):

$$\mathcal{J}_{i+1} := \tilde{\mathcal{I}}_{i+1} \circ \tilde{\mathcal{I}}_i \circ \dots \circ \tilde{\mathcal{I}}_1 : \tilde{\mathcal{H}}_0 \rightarrow \tilde{\mathcal{H}}_{i+1}.$$

and the reconstruction of the air-sea interface can thus be obtained at $i = N-1$.

5. RECONSTRUCTION FOR CONTINUOUS STRATIFICATION

5.1. Overview. We now consider the task of reconstructing the free surface of a *continuously* stratified traveling wave from pressure data. The method of the previous section was to adopt conformal coordinates where the governing equation is simply Laplace in a strip. As continuously stratified waves are rotational, the hodograph transform is no longer available, and the dynamics in the interior are too complicated to permit a similar approach. With that in mind, we will instead proceed by *approximating* the continuous wave by a layer-wise constant density and irrotational one.

Before giving the details of this procedure, let us recall some terminology and important results from the literature. The existence of small-amplitude supercritical solitary waves in both the continuous stratification and layer-wise smooth stratification regimes can be found in Turner [40, 41], where they are built via a limit of periodic solutions with periods increasing to ∞ . Supercritical here refers to the fact that the wave speed is sufficiently large so that the (generalized) Froude number for the flow lies below 1. Turner (and all of the contemporaneous existence theory) dealt exclusively with steady waves for which the Bernoulli function β has the form (2.12h). As we have discussed previously, this is the only possibility if one considers solitary waves that have uniform velocity upstream and downstream. Because he aimed to construct such waves as limits of periodic solutions, Turner naturally looked for periodic waves that also have this type of Bernoulli function. It will be important for our later analysis to note that both the periodic and solitary waves in Turner's theory are waves of elevation, that is, their streamlines lie above those for the corresponding laminar flow. More precisely, if we define \mathring{h} as in (2.12e), then a wave of elevation is one whose height function h satisfies

$$(5.1) \quad h - \mathring{h} > 0 \quad \text{in } \overline{\mathcal{R}} \setminus \{p = p_0\}.$$

Our reconstruction method is based upon the fact that small-amplitude periodic waves with β as in (2.12h) depend continuously on the streamline density function. This was proved recently by the authors in [8]. Briefly, the result is the following.

Theorem 5.1 (Chen–Walsh [8]). *Fix a Hölder exponent $\alpha \in (0, 1)$, a period L sufficiently large, a depth $d > 0$, and a pseudo volumetric mass flux $p_0 < 0$. Let $\rho_* \in C^{1,\alpha}([p_0, 0])$ with $\rho_*(0) = 1$ be a stable streamline density function and suppose that h_* is a corresponding solution of (2.14) for a wave speed c_* that is supercritical. If h_* is a wave of elevation and sufficiently small-amplitude, then the following statements hold.*

- (A1) *There is a neighborhood \mathcal{U} of ρ_* in $L^\infty([p_0, 0])$ such that for any $\rho \in \mathcal{U}$ with $\rho(0) = 1$ that is non-increasing and piecewise smooth, there is a solution (h, ρ, c) to (2.14) such that h is a strict wave of elevation and c is supercritical.*
- (A2) *We have the estimates*

$$(5.2) \quad \|h - h_*\|_{W_{per}^{1,r}(\overline{\mathcal{R}})} \leq C_1 \|\rho - \rho_*\|_{L^\infty}$$

and,

$$(5.3) \quad c - c_* = \frac{1}{d} \int_{p_0}^0 \left[\frac{1}{\sqrt{\rho}} - \frac{1}{\sqrt{\rho_*}} \right] ds = \mathcal{O}(\|\rho - \rho_*\|_{L^\infty}).$$

for some constant C_1 depending on ρ_* , h_* , and L . Here $r := 2/(1 - \alpha)$ is chosen so that $W^{1,r}$ embeds continuously in $C^{0,\alpha}$.

- (A4) *Moreover, the pressure trace on the bed converges in the following sense. Let a connected set $I \subset \subset [p_0, 0] \setminus \{p_1, \dots, p_{N-1}\}$ be given with $p_0 \in I$, and assume that $\rho \in C^{1,\alpha}(\overline{I})$. Denote by P_{bot} the trace of the pressure on the ocean bed for the traveling wave with density ρ , and let P_{bot*} be the trace of P_* on the bed. Then*

$$(5.4) \quad \|P_{bot} - P_{bot*}\|_{C_{per}^{0,\alpha}(\mathbb{R})} \leq C_2 \left(\|\rho - \rho_*\|_{L^\infty([p_0, 0])} + \|\rho - \rho_*\|_{C^{1,\alpha}(\overline{I})} \right),$$

where $C_2 > 0$ depends on the length of I , ρ_* , h_* , and L .

The basic structure of the approximation scheme is intuitive, but several nontrivial technical complications arise. Take a solitary wave with continuous stratification that is small-amplitude, supercritical, and a wave of elevation. Assume, as in Turner, that this wave can be realized as the limit of periodic solutions of increasing period. Each of these periodic waves can in turn be approximated by a layer-wise constant density irrotational flow using Theorem 5.1. Finally, the multi-layered flows can be reconstructed from the pressure data as detailed in Section 4.

5.2. Layer-wise approximation to continuous stratification. In order to rigorously implement the approach suggested above, however, we need to more closely examine the convergence of periodic waves to solitary waves. Unfortunately, this also requires working in a new coordinate system that can be seen as a normalized version of the semi-Lagrangian variables. Specifically, we wish to non-dimensionalize and rescale the streamline coordinate p by mapping

$$(q, p) \mapsto (\xi, \zeta), \quad \text{where} \quad \xi := \frac{q}{d}, \quad \zeta := \frac{1}{cd} \int_0^p \frac{1}{\sqrt{\rho(s)}} ds.$$

For each $L > 0$, the rectangle $\mathcal{R}^L := \{(q, p) \in (-L, L) \times (p_0, 0)\}$ is then taken to

$$\mathcal{S}^L := \{(\xi, \zeta) : \xi \in (-\frac{L}{d}, \frac{L}{d}), \zeta \in (-1, 0)\},$$

with each layer \mathcal{R}_i^L being sent to

$$\mathcal{S}_i^L := \{(\xi, \zeta) : \xi \in (-\frac{L}{d}, \frac{L}{d}), \zeta \in (\zeta_{i-1}, \zeta_i)\}, \quad \text{where } \zeta_i := \frac{1}{cd} \int_0^{p_i} \frac{1}{\sqrt{\rho(s)}} ds.$$

We will continue to denote the full strip by \mathcal{R} . Likewise, \mathcal{R}_i , \mathcal{S} , and \mathcal{S}_i are each unbounded.

The height equation (2.14) is then recast in terms of a new unknown

$$(5.5) \quad w = w(\xi, \zeta) := \frac{y(\xi, p(\zeta))}{d} - \zeta,$$

which will satisfy another quasilinear elliptic problem with the rescaled streamline density function

$$(5.6) \quad \hat{\rho}(\zeta) := \rho(p(\zeta))$$

appearing as a coefficient. Here, w measures the deviation of the streamlines from their height at infinity. One advantage of using this quantity is that it decays upstream and downstream and thus lies in a Sobolev space set on \mathcal{S} . Note that w is a wave of elevation (5.1) provided that

$$(5.7) \quad w > 0 \quad \text{in } \overline{\mathcal{S}} \setminus \{\zeta = -1\}.$$

Following Turner, to w as in (5.5), we associate an energy over the set $\mathcal{K} \subset \mathcal{S}$

$$(5.8) \quad F(w, \mathcal{K}) := \int_{\mathcal{K}} \hat{\rho} \frac{|\nabla w|^2}{1 + w_\zeta} d\xi d\zeta.$$

Note that the absence of stagnation (2.3d) requirement translates to the statement that $1 + w_\zeta > 0$ in these variables, hence the above integral is well-defined. Physically, F describes the amount of kinetic energy imparted by the deviation from the far field state.

Let us now more precisely formulate the class of solitary waves for which we can successfully prosecute the scheme outlined above.

Assumption 1. Consider a solitary stratified water wave represented by $(w_s, \hat{\rho}, c_s)$. We require that (i) the wave has finite energy

$$F(w_s, \mathcal{S}) = R^2 < \infty,$$

that is sufficiently small; (ii) $\|w_s\|_{W^{1,\infty}(\mathcal{S})} < \delta$ for some $\delta > 0$ sufficiently small; (iii) w_s is a wave of elevation (5.7); and (iv) w_s can be realized as a limit of periodic stratified waves with the same energy. This last statement means that there exists a sequence $\{(w_m, \hat{\rho}, c_m)\}$ of continuously stratified periodic steady waves with $w_m, \partial_\xi w_m$ uniformly exponentially localized,

$$\|\nabla w_m\|_{W^{1,\infty}(\mathcal{S}^m)} < \delta, \quad F(w_m, \mathcal{S}^m) = R^2,$$

such that there is a subsequence converging to w_s in $C^{0,\alpha}(\mathcal{S}^k)$ for all $k \in \mathbb{N}$ and $c_m \rightarrow c$.

From Turner [41], we know that waves satisfying Assumption 1 exist. Indeed, much of the solitary wave existence theory for stratified waves is built on exactly this type of limiting construction. Nonetheless, one cannot say that every solitary stratified wave falls within this class, hence we must make it a hypothesis.

Now let $(w_s, \dot{\rho}, c_s)$ be given as above with $\{(w_m, \dot{\rho}, c_m)\}$ the corresponding sequence of continuously stratified periodic waves. Then we may apply Theorem 5.1 to approximate each $(w_m, \dot{\rho}, c_m)$ by a layer-wise constant density irrotational wave $(w_m^N, \dot{\rho}^N, c_m^N)$. From (5.2)–(5.3) we have the estimates

$$\|w_m^N - w_m\|_{C_{\text{per}}^{0,\alpha}(\overline{\mathcal{R}^m})} \leq C \|w_m^N - w_m\|_{W_{\text{per}}^{1,r}(\mathcal{R}^m)} \leq C_1 \|\dot{\rho}^N - \dot{\rho}\|_{L^\infty},$$

and

$$|c_m^N - c_m| \leq C \|\dot{\rho}^N - \dot{\rho}\|_{L^\infty},$$

where $r = 2/(1 - \alpha)$ and $C_1 = C_1(w_m, \dot{\rho}, m)$. Note that using (5.3), we can adjust $\dot{\rho}^N$ to ensure that $c_m^N = c_m$, and thus the multi-layer solutions are $(w_m^N, \dot{\rho}^N, c_m)$.

The difference in the energies can then be estimated:

$$\begin{aligned} |F(w_m^N, \mathcal{S}^m) - F(w_m, \mathcal{S}^m)| &\leq \int_{\mathcal{S}^m} \left| \dot{\rho}^N \frac{|\nabla w_m^N|^2}{1 + \partial_\zeta w_m^N} - \dot{\rho}_s \frac{|\nabla w_m|^2}{1 + \partial_\zeta w_m} \right| d\xi d\zeta \\ &\leq C_2 \|\dot{\rho}^N - \dot{\rho}_s\|_{L^\infty} + C_3 \|\dot{\rho}_s\|_{L^\infty} \|w_m^N - w_m\|_{W^{1,2}(\mathcal{S}^m)} \\ &\leq (C_2 + C_1 C_3 \|\dot{\rho}_s\|_{L^\infty}) \|\dot{\rho}^N - \dot{\rho}_s\|_{L^\infty}, \end{aligned}$$

where $C_2 = C_2(\|\nabla w_m^N\|_{L^\infty(\mathcal{S}^m)}, \|\nabla w_m^N\|_{L^2(\mathcal{S}^m)})$ and $C_3 = C_3(\|\nabla w_m^N\|_{L^\infty(\mathcal{S}^m)}, \|\nabla w_m\|_{L^\infty(\mathcal{S}^m)})$. From [8, Lemma 3.7, Lemma 3.10] we know that,

$$\|\nabla w_m^N\|_{L^\infty(\mathcal{S}^m)}, \|\nabla w_m^N\|_{L^2(\mathcal{S}^m)} \leq C(\|\nabla w_m\|_{L^2(\mathcal{S}^m)}) \leq C(R, \delta)$$

uniformly in N . The exponential localization implies moreover that these constants are uniform in m as well. Therefore

$$(5.9) \quad |F(w_m^N, \mathcal{S}^m) - R^2| \leq C \|\dot{\rho}^N - \dot{\rho}\|_{L^\infty}.$$

Now for any fixed N , following the diagonalization technique as in Turner [40], modulo a subsequence, we have

$$w_m^N \rightarrow w_s^N, \quad \text{as } m \rightarrow \infty, \quad \text{in } C^{0,\alpha}(\mathcal{R}^k) \text{ for all } k \in \mathbb{N},$$

where $(w_s^N, \dot{\rho}, c_s^N)$ is a layer-wise smooth solitary wave carrying an energy

$$F(w_s^N, \mathcal{S}) = \lim_{m \rightarrow \infty} F(w_m^N, \mathcal{S}^m).$$

The convergence of the right-hand side is a consequence of the uniform (in m) exponential decay of w_m^N and ∇w_m^N . From (5.9) we know that

$$|F(w_s^N, \mathcal{S}) - F(w_s, \mathcal{S})| \leq C \|\dot{\rho}^N - \dot{\rho}\|_{L^\infty}.$$

Passing to a further subsequence ensures the convergence of $c_m^N = c_m$ to some c_s^N . So, for any $\epsilon > 0$, taking m large enough gives

$$\begin{aligned} \|w_s^N - w_s\|_{C^{0,\alpha}(\overline{\mathcal{R}^k})} &\leq \|w_m^N - w_s^N\|_{C^{0,\alpha}(\overline{\mathcal{R}^k})} + \|w_m^N - w_m\|_{C^{0,\alpha}(\overline{\mathcal{R}^k})} + \|w_m - w_s\|_{C^{0,\alpha}(\overline{\mathcal{R}^k})} \\ &< 2\epsilon + C \|\dot{\rho}^N - \dot{\rho}_s\|_{L^\infty}, \end{aligned}$$

for all $k \in \mathbb{N}$. Finally, letting $m \rightarrow \infty$ we conclude

$$(5.10) \quad \|w_s^N - w_s\|_{C^{0,\alpha}(\overline{\mathcal{R}^k})} \leq C \|\dot{\rho}^N - \dot{\rho}\|_{L^\infty}, \quad \text{for all } k \in \mathbb{N}.$$

Therefore, given the pressure data for w_s^N , we can apply the reconstruction procedure of the previous subsection to recover w_s up to $\mathcal{O}(\|\dot{\rho}^N - \dot{\rho}\|_{L^\infty})$ in $C^{0,\alpha}$.

5.3. Convergence of the ocean bed parameterization. The argument above shows that we can approximate the continuously stratified solitary wave via a sequence of layer-wise constant density irrotational solitary waves. However, its main weakness is that it requires the pressure data not for the continuously stratified solitary wave (which is observed), but for a limiting (sub)sequence which are only known to exist mathematically. This is partially ameliorated by (5.4) which guarantees that the pressure traces of the subsequence converge to that of the continuous solitary wave.

But there are two issues remaining that must be addressed. The first of these is rather benign: in Section 4, we begin the reconstruction process with the data \mathbf{p} . It is therefore necessary to confirm that the convergence of the pressure likewise extends to this quantity. Note that because \mathbf{p} is defined in conformal variables — which do not have an obvious meaning for continuously stratified flows — it is not immediately clear that this should be true. The second more serious issue is the sensitivity of the reconstruction scheme to noise in the pressure data. This we discuss in the next subsection.

Let h be the height function in semi-Lagrangian coordinates for a steady wave localized near the crest (it can be solitary or periodic). Suppose that, as above, there is a sequence $\{h^N\}$ of layer-wise constant density steady wave localized near the crest that converges to h . Then for each h^N , the conformal coordinates are well-defined and we have

$$t = -\frac{1}{c}\varphi^N(x_0^N(t), -d),$$

and hence

$$-c = \varphi_x^N(x_0^N(t), -d)\dot{x}_0^N(t) = -\psi_y(x_0^N(t), -d)\dot{x}_0^N(t).$$

But then, using the change of variables formulas, this implies that x_0^N solves the ODE

$$\dot{x}_0^N = h_p^N(x_0^N, p_0), \quad x_0^N(0) = 0.$$

In view of this, we define $x_0 = x_0(t)$ to be the solution of

$$(5.11) \quad \dot{x}_0(t) = h_p(x_0(t), p_0), \quad x_0(0) = 0,$$

where the dot denotes differentiation with respect to t . Formally, the initial value problem (5.11) is the limit of the ODE satisfied by x_0^N as $N \rightarrow \infty$. We hope to show that x_0^N likewise converges to x_0 , which then implies \mathbf{p}^N approaches \mathbf{p} .

Proposition 5.1. *Let $\{\rho^N\} \subset L^\infty([p_0, 0])$ be a sequence of piecewise constant stable streamline density functions, with $\rho^N \rightarrow \rho$ in L^∞ , where $\rho \in C^{1,\alpha}([p_0, 0])$ is a stable smooth streamline density function that is constant on some neighborhood I of the bed. Let h and h^N be the height functions for a traveling wave localized near the crest with period $L \in (0, \infty]$, the speed c and c^N , and streamline density function ρ and ρ^N respectively. Assume that*

$$c\sqrt{\rho(p_0)} = c^N\sqrt{\rho^N(p_0)},$$

and that there exist constants $M, \gamma > 0$ independent of L and N such that

$$|h_p(q, p_0) - \dot{h}_p(p_0)|, |h_p^N(q, p_0) - \dot{h}_p^N(p_0)| \leq Me^{-\gamma|q|}.$$

If $L = \infty$, i.e., the waves are solitary, also suppose that $h_p^N(\cdot, p_0) \rightarrow h_p(\cdot, p_0)$ in $L^\infty(\mathbb{R})$. Then it follows that

$$x_0^N \rightarrow x_0 \quad \text{in } L^\infty(\mathbb{R}).$$

Proof. Denote $z^N := x_0 - x_0^N$, so that z^N satisfies

$$(5.12) \quad z^N(t) = \int_0^t [h_p(x_0(\tau), p_0) - h_p^N(x_0^N(\tau), p_0)] d\tau.$$

The no stagnation condition implies that there exists $\sigma > 0$ such that

$$h_p(\cdot, p_0), h_p^N(\cdot, p_0) \geq \sigma > 0, \text{ for all } N \geq 1.$$

Recalling (5.11), this guarantees that

$$x_0(t), x_0^N(t) \geq \sigma t, \quad \text{for all } t \in [0, \infty).$$

Fix $\epsilon > 0$, and choose $T > 0$ sufficiently large so that

$$\int_T^\infty e^{-\sigma\gamma t} dt < \frac{\epsilon}{4M}.$$

Starting from (5.12), we write

$$\begin{aligned} \|z^N\|_{L^\infty} &\leq \int_0^T |h_p(x_0(\tau), p_0) - h_p^N(x_0^N(\tau), p_0)| d\tau + \int_T^\infty [h_p(x_0(\tau), p_0) - h_p^N(x_0^N(\tau), p_0)] d\tau \\ &=: \mathbf{I} + \mathbf{II}. \end{aligned}$$

To estimate **II**, we exploit the fact that $\dot{h}_p(p_0) = \dot{h}_p^N(p_0)$, and the exponential localization:

$$\begin{aligned} \mathbf{II} &\leq \int_T^\infty |h_p(x_0(\tau), p_0) - \dot{h}_p(p_0)| d\tau + \int_T^\infty |h_p^N(x_0^N(\tau), p_0) - \dot{h}_p^N(p_0)| d\tau \\ &\leq 2M \int_T^\infty e^{-\gamma\sigma t} dt < \frac{\epsilon}{2}. \end{aligned}$$

On the other hand, for **I** we use the simple estimate

$$\mathbf{I} \leq T \|h_p(\cdot, p_0) - h_p^N(\cdot, p_0)\|_{L^\infty} \leq T \|h_p(\cdot, p_0) - h_p^N(\cdot, p_0)\|_{L^\infty} < \frac{\epsilon}{2},$$

for N sufficiently large. Observe that in the periodic case $L < \infty$, we have

$$\|h_p(\cdot, p_0) - h_p^N(\cdot, p_0)\|_{L^\infty} \leq C (\|\rho^N - \rho\|_{L^\infty([p_0, 0])} + \|\rho^N - \rho\|_{C^{1,\alpha}(I)}),$$

and hence the L^∞ convergence of h_p^N on the bed is implied by our previous assumptions.

The argument above gives the convergence of x_0^N to x_0 in $L^\infty([0, \infty))$; the argument for $L^\infty((-\infty, 0])$ is similar. \square

5.4. Pressure noise. Let us now discuss the sensitivity of the above considerations to pressure data. First, it is important to note that any scheme that is attempting to reconstruct from pressure must come to terms with the physical limitations of pressure transducers. It is of course impossible to perfectly read the pressure, and in particular, existing instruments fail to capture very high frequency components. Likewise, computers are finite precision machines, and so any practical implementation of the recovery scheme will necessitate additional approximation. For instance, it is common to use a windowed transfer functions that simply ignores the high frequency range above a certain cut-off. Mathematically, this procedure corresponds to applying a Littlewood-Paley projection onto a finite frequency ball. However, because we are working with an ill-posed PDE system, it is very challenging to prove that convergence occurs as the cut-off radius is increased.

This seems endemic to all reconstruction from pressure procedures, even for constant density irrotational flows. Nonetheless, actual numerical experiments suggests that the schemes converge, and give reasonable agreement with observation [14, 31].

Unfortunately, the reconstruction process for continuously stratified fluids given above possesses an additional source of error. Observe that in our approximation we take the pressure reading of a solitary wave with continuous stratification and treat this as pressure data for a layer-wise constant density irrotational wave. But, the unique determinability result indicates that the pressure trace for these two waves cannot possibly coincide. We can therefore think of this as attempting to run the reconstruction scheme of Section 4 with noisy pressure data.

To be more concrete, denote

$$(5.13) \quad P_{\text{bot}}^N = P_{\text{bot}*} + P_{\text{err}},$$

where $P_{\text{bot}*}$ is the bottom pressure reading of the solitary wave with continuous stratification, and P_{bot}^N is the trace of the pressure for the N -layered solitary wave. From (4.3) we then have

$$(5.14) \quad \mathbf{p}^N = \mathbf{p} + \frac{P_{\text{err}}}{\varrho_1} + \frac{1}{\varrho_1} \int_{p_0}^0 g [\rho(p) - \rho^N(p)] dp =: \mathbf{p} + \mathbf{p}_{\text{err}}.$$

In order to recover the true N -layered flow, one should use $\mathbf{p}^N(t)$ in the iteration scheme (4.36), rather than $\mathbf{p}(t)$. Hence the error from our iteration scheme consists of two parts: one from the limiting process by taking large number of layers; the other from the pressure error.

Let

$$f(\mathbf{p}) = \frac{1}{\sqrt{\varrho_1}} \left(\frac{c}{\sqrt{c^2 - 2\mathbf{p}}} - 1 \right).$$

From (4.24) the error in the reconstruction of the first interface is

$$(5.15) \quad Y_1^N - Y_1 = \mathcal{F}^{-1} \left\{ \frac{\sinh[(s_1 - s_0)\xi]}{\xi} \mathcal{F} [f(\mathbf{p}^N) - f(\mathbf{p})] (\xi) \right\}.$$

The difference term can be computed as

$$f(\mathbf{p}^N) - f(\mathbf{p}) = \frac{2c}{\sqrt{\varrho_1}(c^2 - 2\mathbf{p})(c^2 - 2\mathbf{p}^N)} \left(\sqrt{c^2 - 2\mathbf{p}} + \sqrt{c^2 - 2\mathbf{p}^N} \right) \cdot (\mathbf{p} - \mathbf{p}^N)$$

To justify the inverse Fourier transform, or more precisely, to make sense of Y_1^N and Y_1 as functions, one needs to ensure that $\mathcal{F}[f(\mathbf{p}^N)]$ and $\mathcal{F}[f(\mathbf{p})]$ have sufficient decay, e.g.

$$e^{(s_1-s_0)|\xi|} \mathcal{F}[f(\mathbf{p}^N)](\xi), \quad e^{(s_1-s_0)|\xi|} \mathcal{F}[f(\mathbf{p})](\xi) \in L^2(\mathbb{R}).$$

In fact, this can be proved rigorously if the continuously stratified wave has constant density in \mathcal{S}_1 . Following Clamond-Constantin [10], we see that the pressure transfer function $\mathbf{p}^N(t)$ can be holomorphically extended to a function $\mathbf{p}^N(z)$ with $z = t + is$ defined in the strip $\{(t, s) \in \mathbb{R} \times (2s_0 - s_1 - \varepsilon, s_1 + \varepsilon)\}$ for some $\varepsilon > 0$. Likewise, the analyticity of $\mathbf{p}^N(z)$ and the no stagnation condition imply that the function

$$f(\mathbf{p}^N(z)) = \frac{1}{\sqrt{\varrho_1}} \left(\frac{c}{\sqrt{c^2 - 2\mathbf{p}^N(z)}} - 1 \right)$$

is analytic in $\mathbb{R} \times [-\lambda, \mu]$, where $\lambda = \mu = s_1 - s_0 + \frac{\varepsilon}{2}$. Thus, the Paley–Weinder Theorem (cf., e.g. [32, Theorem 4]) implies that

$$e^{(s_1-s_0)|\xi|} \mathcal{F}[f(\mathbf{p}^N)](\xi) \in L^2(\mathbb{R}).$$

However the above argument does not extend beyond the region of constant density, and hence the decay of $\mathcal{F}[f(\mathbf{p})](\xi)$ becomes very hard to verify for more general stratification. Worse still, because one loses analyticity crossing the internal interfaces, it is unclear how this reasoning could be used in any layer but the first. For this reason, we cannot currently prove that our method represents a convergent numerical scheme. On the other hand, as it does succeed an analytical approximation method, we anticipate that it will have applications for studying the qualitative behavior features of continuously stratified solitary waves.

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